Local Similarity Solution of Laminar Forced-Convection in Entrance Region for Flow between Two Parallel Plates

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الحل المتشابهي المحلي للحمل الجبري الرقائقي في منطقة
الدخول للسريان بين لوحين متوازيين

خلاصة - في هذا البحث تم دراسة النمو الآني لكل من الطبقة المحددة للسرعة والطبقة المحددة الحرارية لحالة الحمل الجبري للسريان بين لوحين متوازيين. باستخدام طريقة الحل المتشابهي المحلي أمكن تحويل كل من معادلة الحركة في اتجاه السريان ومعادلة الطاقة الى معادلات تفاضلية عادية , حيث حلت عدديا بطريقة رانج-كتا للمعادلات التفاضلية العادية مع استخدام طريقة التصويب لحل مسائل القيم الحدية. وبالتعريف المناسب لكل من درجة الحرارة اللابعدية ومعامل انتقال الحرارة أمكن استخدام حل ينطبق على حالة ثبات وتساوي درجة حرارة كل من اللوحين وفي نفس الوقت على حالة ثبات وعدم تساوي درجة حرارة اللوحين. وقد تم حساب رقم نوسلت للقيم المختلفة للبعد من طرف الممر وذلك لقيم رقم برانتل التي تساوي ٥ر٠ , ٠ر١ , ٠ر٢ . ويعتبر هذا المدى لرقم برانتل يغطي أغلب الغازات المستخدمة في التطبيقات الهندسية.

Abstract- The simultaneous development of both hydrodynamic and thermal boundary layers is theoretically examined in case of laminar forced-convection in entrance region for flow between two parallel plates. This is done by the application of the local similarity solution-method. According to this method, momentum and energy equations of the problem are transferred to ordinary differential equations, which are solved numerically by the Runge-Kutta method accompanied with the Shooting method of boundary value problems.

The dimensionless temperature and heat transfer coefficient are suitably defined so that the obtained Nusselt number is valid in both cases of constant and equal wall- temperatures ;and in the same time in case of constant and unequal wall- temperatures. The values of Nusselt number are calculated at different positions along the passage for the value of Prandtl number of 0.5, 1.0& 2.0. This range of Prandtl number satisfies almost the important gases in the engineering applications.

1. INTRODUCTION

In the design of heat exchangers, the prediction of the heat transfer coefficient with a good accuracy is a very important factor. At the inlet of the heat exchangers, both hydrodynamic and thermal boundary layers develop simultaneously. Therefore laminar forced convection solutions in combined entrance region represent an important class of solutions for heat exchangers. Kakac and Yener [1] made a good survey of the previous studies of laminar forced convection in various duct geometries under constant wall temperature and constant wall heat flux boundary conditions, for Newtonian and constant physical properties fluids. One of the simple geometries for the mathematical treatment, is the flow between two parallel plates.

Many different methods have been developed to solve the governing equations of the problem of heat transfer in simultaneous development of velocity and temperature profiles in the entance region. Sparrow [2] is the first investigator who studied the simultaneous development of the velocity and temperature profiles for parallel plate channels. Rohsenow and Choi [3] have graphically represented Sparrow's results for mean Nusselt numbers for Prandtl number of values of 0.0,0.01,0.72,1,2,10,50 and ∞ . An approximate series solution for the combined entrance region under constant wall temperature boundary conditions was obtained by Stephan [4]. Also, an analysis of the simultaneously developing region was made by finite difference method [5,6]. Hwang and Fan [5] reported Nusselt number for Prandtl numbers in range of 0.01 to 50, for constant wall-temperature and constant heat flux boundary conditions. Mercer [6] proposed an empirical relation for mean Nusselt number under constant wall temperature boundary conditions. Miller and Lundberg [7] extended the work of [5,6,7] for boundary conditions of constant but unequal wall temperatures. They used Bodol's velocity distribution [8]. Their results have been presented for Prandtl number range from 0.5 to 10. The present problem was also solved by the integral method [9,10,11]. Naito [9] obtained the solution for velocity entrance region by Karman - Pohlhausen integral method. Subsequently he obtained the solution for the combined entrance region under the constant heat flux boundary condition. Nusselt numbers of this work are in good agreement with the results obtained by Siegel and Sparrow [10]. Bhatti and Savery [11] made a graphical representation of Nusselt number for Prandtl numbers ranging from 0.01 to 10.00.

Similar solution for laminar forced convection heat transfer from single plate was obtained by Lin and Lin [12]. Their similarity solution provides very accurate solutions for laminar forced convection heat transfer from either an isothermal surface or a uniform- flux boundary to a fluid of any Prandtl number. The purpose of the present work is to make a local similarity solution for laminar forced convection heat transfer in entrance region for flow between two parallel plates. The local similarity solution of boundary layer equations was first derived by Sparrow, Quack and Boerner [13]. In present work the achieved local similarity solution is valid for boundary conditions of constant and equal wall temperatures and also for the boundary condition of constant and unequal wall temperatures.

2. BOUNDARY-LAYER EQUATIONS

Consider the laminar boundary-layer flow between two parallel plates as shown in Fig. (1). The velocity of approach, temperature of the fluid at the inlet and the distance between the two plates are denoted as u_o, T_o and b, respectively. The velocity at the axis of similarity is denoted by $u_{o,x}$. The case of constant wall-temperatures(T_1, T_2) will be studied. The wall temperatures can be equal or unequal. Constant fluid properties are assumed.

The governing equations can be written in Cartesian co-ordinate x, y as follows :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad , \tag{1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad , \tag{2}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \quad , \tag{3}$$

where u and v are the velocity components in x- and y-directions, respectively, T is the temperature of fluid, ν the kinematic viscosity and α the thermal diffusivity. In the momentum equation, the effect of pressure variation in x-direction is neglected. The value of the velocity at the axis of similarity $(u_{o,x})$, at any value of x must satisfy the continuity equation in integral form, which states :

$$\int_0^{b/2} u \; dy = \frac{1}{2} u_o \, b \quad . \tag{4}$$

Equations (1)-(3) form a system of equations for three unkowns u, v and T. This system has to satisfy the following boundary conditions :

$$\begin{aligned} u = v = 0 \; , \quad T_w = T_1 \qquad & \text{at} \quad y = 0 \quad ; \\ u = v = 0 \; , \quad T_w = T_2 \qquad & \text{at} \quad y = 0 \quad ; \\ u = u_{o,x} \qquad & \text{at} \quad y = b/2 \quad . \end{aligned} \tag{5}$$

To express the governing equations in dimensionless form, one introduces new independent variables ξ, η as follows :

$$\xi = \sqrt{\frac{u_o x}{\nu}} = \sqrt{Re_x} \quad , \quad \eta = y \sqrt{\frac{u_o}{\nu x}} \quad . \tag{6}$$

Furthermore, a dimensionless stream function and a dimensionless temperature are defined according to the following relations :

$$f(\xi,\eta) = \psi(x,y)/\sqrt{u_o \nu x} \quad , \tag{7}$$

$$\theta(\xi,\eta) = 2(T-T_o) \,/\, (1+\phi)(T_1-T_2) \; ,$$

$$\phi = (T_2-T_o) \,/\, (T_1-T_o) \quad . \tag{8}$$

where $\psi(x,y)$ is the stream function , which is introduced to satisfy the continuity equation (1). The dimensionless temperature (θ) and the wall temperature ratio (ϕ) are defined in such a manner to satisfy the condition of constant and equal wall temperatures, and also the case of constant and unequal wall temperature. In first case ϕ has the value of one and on other hand, in second case the value of ϕ could take a value in the range of nil to one. Substitution of equtions (6)-(8) into equations (1)-(3) and (5) leads to the following dimensionless form of the momentum and energy equations (primes denoting differentiation with respect to η) :

$$f''' + \frac{1}{2} f f'' = 0 \quad , \qquad (9)$$

$$\frac{2}{Pr} \theta'' + f \theta' = 0 \quad , \qquad (10)$$

with the boundary conditions :

$$f = f' = 0 \quad , \quad \theta_1 = \frac{2}{1+\phi} \qquad \text{at } \eta = 0 \; ; \qquad (11a)$$

$$f' = 0 \quad , \quad \theta_2 = \frac{2\phi}{1+\phi} \qquad \text{at } \eta = \eta_b \; ; \qquad (11b)$$

$$f' = \frac{u_{o,x}}{u_o} \qquad \text{at } \eta = \eta_{b/2} \quad , \qquad (11c)$$

where η_b and $\eta_{b/2}$ denote the value of the dimensionless independent variable η corresponding to the total distance between the plates (b) and to the half of this distance (b/2), respectively. According to the local similarity solution method [13], the derivatives of the variables f and θ with respect to ξ in equations (9)-(10) are ignored, and ξ is dealt with as a parameter, which is varied according to the value of x. Moreover, according to the definitions of ξ, η and f [equations(6)-(7)], and equation (4), the proper value of $f'(\xi, \eta_{b/2})$ must satisfy the following equation :

$$\eta_{b/2} = \int_0^{\eta_{b/2}} f' \, d\eta \quad . \qquad (12)$$

Equations (9)-(11) represent a system of ordinary differential equations with their boundary conditions, which must be satisfied. Because of the similarity of the velocity profile about the similarity-axis of the passage, it is convenient to solve the momentum equation for the half of flow field (from y=0 to y=b/2). The value of η has a fixed value of nil at the boundary, where y=0. But from equation (6), the value of η at the other boundary (at y=b/2) depends upon the value of x and, in turn, upon the value of ξ, as it is clear from the following equation :

$$\eta_{b/2} = \frac{b}{2} \sqrt{\frac{u_o}{x \nu}} = \frac{1}{2} \frac{b u_o}{\nu} \frac{1}{\xi} \quad . \qquad (13)$$

3. NUMERICAL PROCEDURE

According to the local similarity-method [13], the system of equations (9)-(11) is solved for different values of the parameter ξ and hence for different values of η_b. First, the momentum equation (9) and its boundary conditions [equations (11a)-(11c)] is solved, numerically, by the Runge-Kutta method of ordinary differential equations. The value of f' at $\eta=\eta_{b/2}$ is justified to the proper value, using the shooting method of the boundary value problems accompanied with the Newton-Raphson method of non-algebric equations. This proper value of f' produces a velocity profile, which satisfies equation (12). Knowing the velocity profile, one can proceed to solve the energy equation (10) and its boundary conditions [equations (11a)-(11b)]. With the same numerical procedure used to solve the momentum equation, the energy equation is solved. The solution of it consists of two parts, the frist is the solution of the energy equation from the boundary, where $\eta=0$ and the second starts from the other boundary, where $\eta=\eta_b$. The two parts of solution is properly coupled at the position, where the temperature of the fluid has the minimum value ($\theta=0$). A constant step size $\Delta\eta$ is taken as 0.1. The highest value of η_b, used to obtain the present results, is taken as 50 and hence the corresponding value of ξ/Re_b has to be 0.02 (smallest value of ξ). The value of the parameter ξ is increased by a variable interval $\Delta\xi$ corresponding to a constant interval $\Delta\eta_b=-2$.

When the velocity and temperature fields have been obtained the local Nusselt number Nu_x ,and the local coefficient of friction C_f can be determined according to the following definitions :

$$Nu_x = \frac{h\,x}{k} , \qquad C_f = \tau_w / \rho u_o^2 , \tag{14}$$

where, the shear stress at the wall τ_w, the local heat transfer coefficient h and the thermal coductivity k are determined according to the following equations :

$$\tau_w = \rho \nu \left(\frac{\partial u}{\partial y} \right)_{y=0} , \tag{15}$$

$$q_w = h \left(\frac{T_1 + T_2}{2} - T_m \right)$$

$$= k \left(\left| \left(\frac{\partial T}{\partial y} \right)_{y=0} \right| + \left| \left(\frac{\partial T}{\partial y} \right)_{y=b} \right| \right) . \tag{16}$$

Introducing dimensionless variables in equations (14)-(16), one obtains the following expressions of local Nusselt number and local coefficient of friction :

$$Nu_x / \sqrt{Re_x} = \left(|\theta'(0,\xi)| + |\theta'(\eta_b,\xi)| \right) / (1 - \theta_m) , \tag{17}$$

$$C_f \sqrt{Re_x} = f''(\xi,0) , \tag{18}$$

where Re_x denotes the local Reynolds number ($u_o x/\nu$) and θ_m is the average dimensionless temperature.

4. RESULTS AND DISCUSSION

The numerical obtained results are represented in the following figures. Fig. (2) represents the velocity profile at different values of η_b and hence at different values of ξ. At

η_b=10 ($\sqrt{Re_x}/Re_b$ = 0.1) the profile takes a shape near that of the fully developed flow (parabolic profile). Figures (3)-(4) are the dimensionless temperature profile for the cases of constant and equal wall-temperatures (ϕ = 1.0), and constant and unequal wall-temperatures (ϕ = 0.25). Fig. (3) is drawn for Pr = 2.0 at values of η_b= 40, 30, 20 & 10, or in another word at values of $\sqrt{Re_x}/Re_b$ = 0.025, 0.033, 0.05& 0.1. Fig. (4) shows the developing of temperature profile in case of ϕ = 0.25 (θ_1=1.6, θ_2=0.4). The dimensionless bulk temperature (θ_m) along the passage at different Prandtl number is given in Fig. (5). Local Nusselt number ($Nu_x/\sqrt{Re_x}$) against the dimensionless distance ($\sqrt{Re_x}/Re_b$) along the passage for Pr =2.0, 1.0, 0.5 is given in Fig. (6). For the purpose of comparison with the results of previous investigations, the local Nusselt number (Nu_b = hb / k) is represented against dimensionless distance x^* (x^* =(x/b)/(Re_xPr)) in Fig. (7). It is clear that, the value of Nusselt number (Nu_b) takes an asymptotic value of about twelve for all values of Prandtl numbers. This asymptotic value is lower than that obtained in literatures ($Nu_{b/2}$= 7.504). The reason of this deviation seems to be due to the negligence of the effect of the pressure variation along the passage. The numerical results of Nu_x and Nu_b are tabulated in tables (1)-(2). Fig. (8) shows maximum velocity, local Nusselt number and coefficient of friction for Prandtl number of one against the dimensionless distance $\sqrt{Re_x}/Re_b$.

5. CONCLUSION

The present work proves that, the local similarity solution method is suitable to solve the governing equations of the flow not only over a single surface, but also for flow through conduits,the simplest shape of them is the flow between two parallel plates. The computer program based upon the present derived solution is self-starting one. More than the boundary conditions of the problem, no further informations are required to carry out the calculations at any position along the passage. The obtained Nusselt number, in present study, is valid for case of constant and equal wall-temperatures, and at the same time for the case of constant and unequal wall-temperatures. The deviation of the numerical results from that of previous works is, probably, due to the negligence of the effect of pressure variations inside the passage. Hence, it is proposed to make more studies to examine the correctness of this suggested reason.

6. NOMENCLATURE

b	the normal distance between the two plates
C_f	coefficient of friction, $\tau_w/\rho u_o$
f	dimensionless stream function, $\psi/\sqrt{(u_o \nu x)}$

h	local heat transfer coefficient, defined through eq.(16)
k	thermal conductivity
Nu_b	local Nusselt number based upon b, hb/k
Nu_x	local Nusselt number, hx/k
Pr	Prandtl number, ν/α
q_w	heat flux at the wall
Re_b	local Reynolds number based upon b, $u_o b/\nu$
Re_x	local Reynolds number, $u_o x/\nu$
T	temperature
T_o	temperature of fluid at the inlet cross-section
T_1, T_2	temperature of lower and upper wall ,respectively
T_m	bluk temperature
T_w	wall-temperature
u	velocity component in x-direction
u_o	the velocity at inlet cross-section of the passage
$u_{o,x}$	the velocity at the axis of similarity
v	velocity component in y-direction
x	co-ordinate along the lower wall of the passage
x^*	dimensionless distance, $(x/b)/(Re_b Pr)$
y	co-ordinate normal to the lower wall of passage
α	thermal diffusivity, $\rho C_p/k$
η	dimensionless independent variable, $y.\sqrt{(u_o/x\nu)}$
η_b	the value of η at the upper wall, $Re_b/\sqrt{Re_x}$
$\eta_{b/2}$	the value of η at the half of the distance between the two walls
ξ	dimensionless independent variable, $\sqrt{(u_o x)/\nu}$
ϕ	wall-temperatures ratio, $(T_2-T_o)/(T_1-T_o)$
ν	kinematic viscosity
ρ	density
θ	dimensionless temperature, $2\ (T-T_o)\ /\ (1+\phi)(T_1-T_o)$
τ_w	wall shear stress,. $\rho\nu\ (\frac{\partial u}{\partial y})_{y=0}$
ψ	stream function

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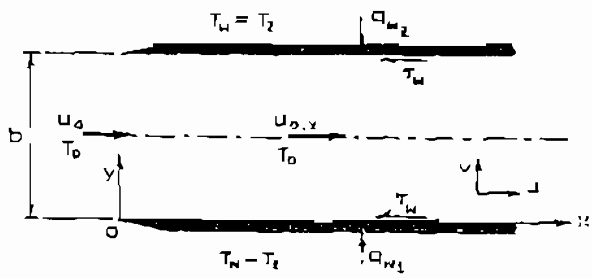

Fig. (1) Schmatic descripticn of the flow between two parallel plates .

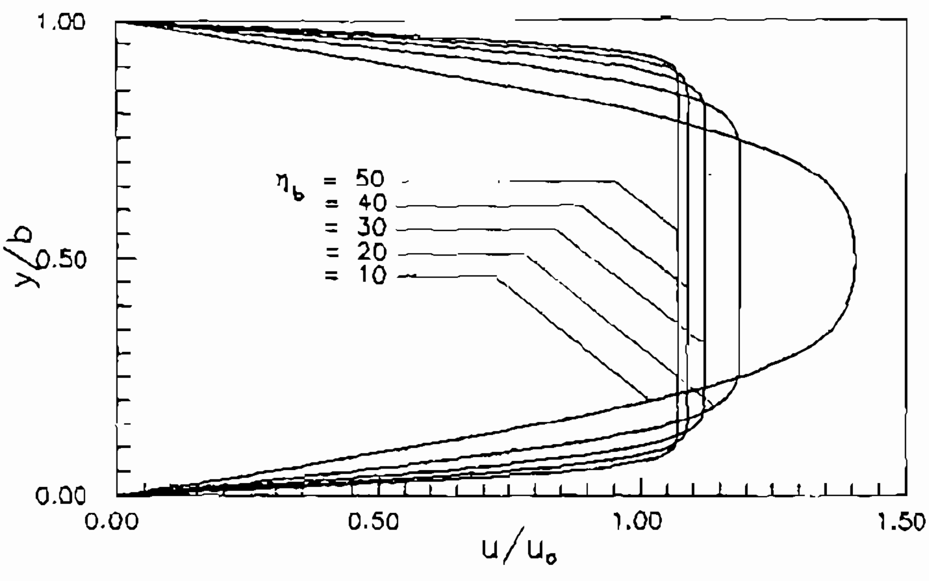

Fig. (2) The development of the velocity profile of laminar flow in the entrance region of the passage.

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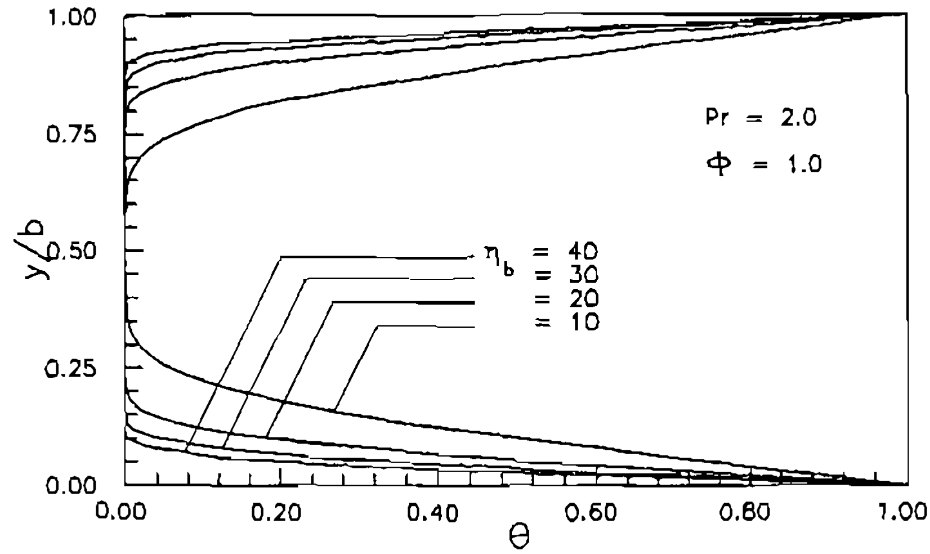

Fig. (3) The development of the temperature profile in the thermal entrance region of the passage in case of constant and equal wall temperatures .

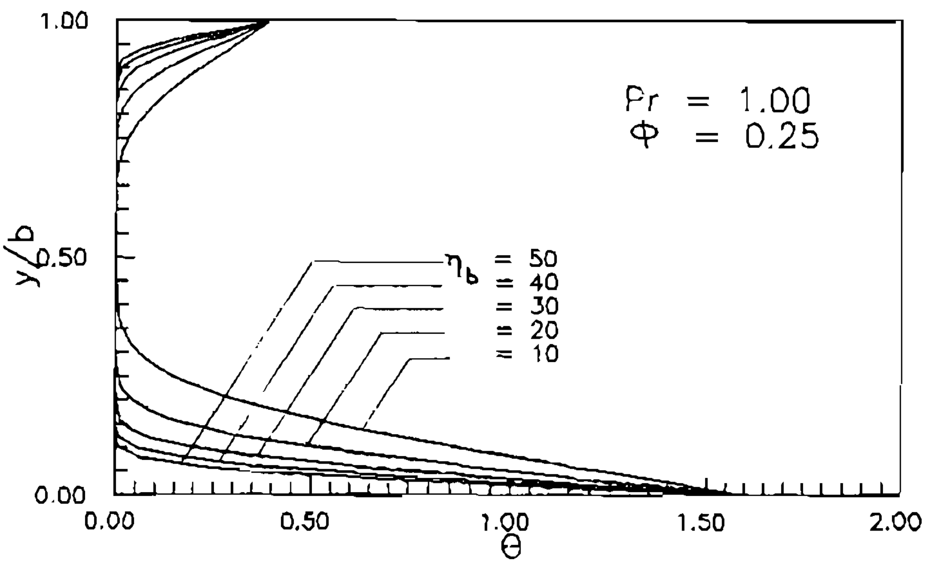

Fig. (4) The development of temperature profile in thermal entrance zone of the passage in case of constant difference of wall temperatures (Φ = 0.25) .

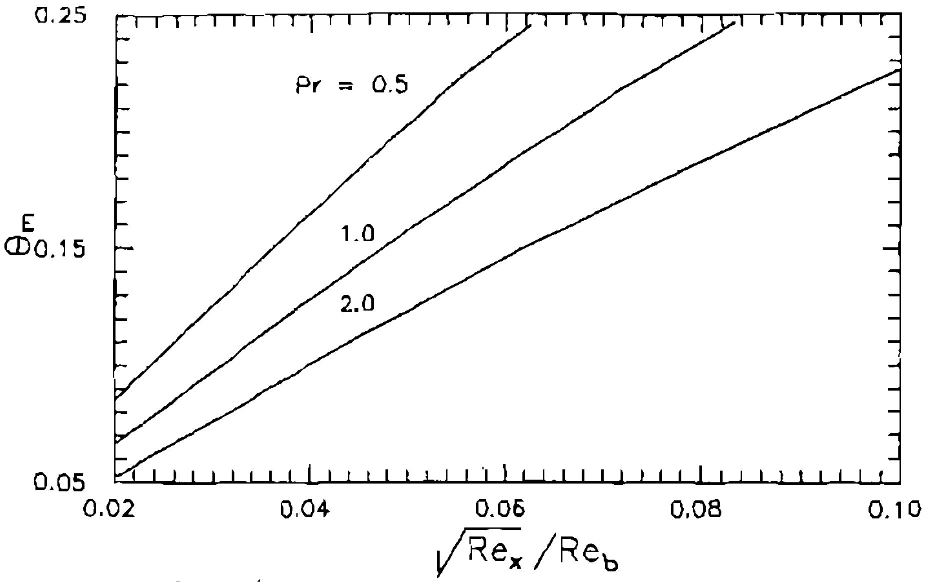

Fig. (5) The average temperc:ure in thermal entrance zone in case of constant wall temperature .

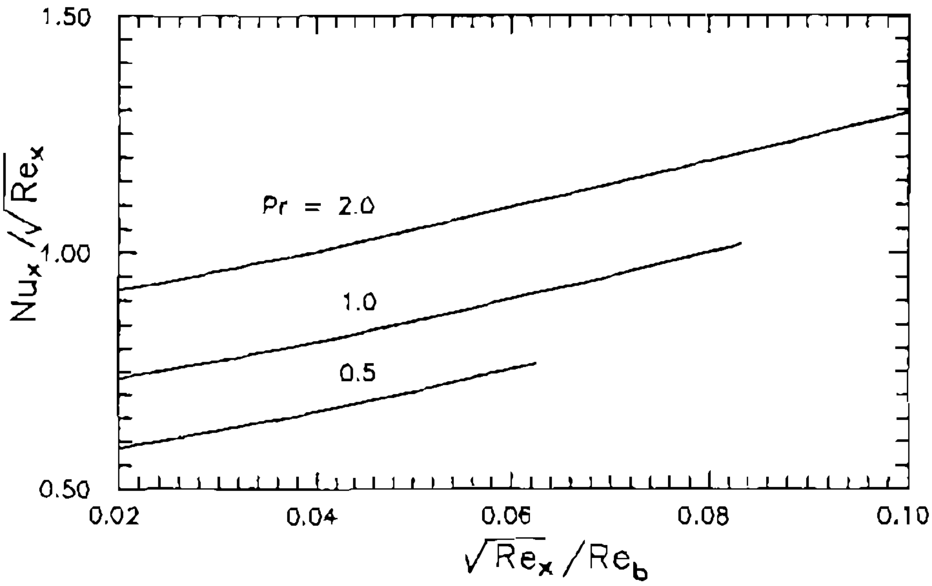

Fig. (6) Local Nusselt number in thermal entrance zone in case of constont wall temoeroture .

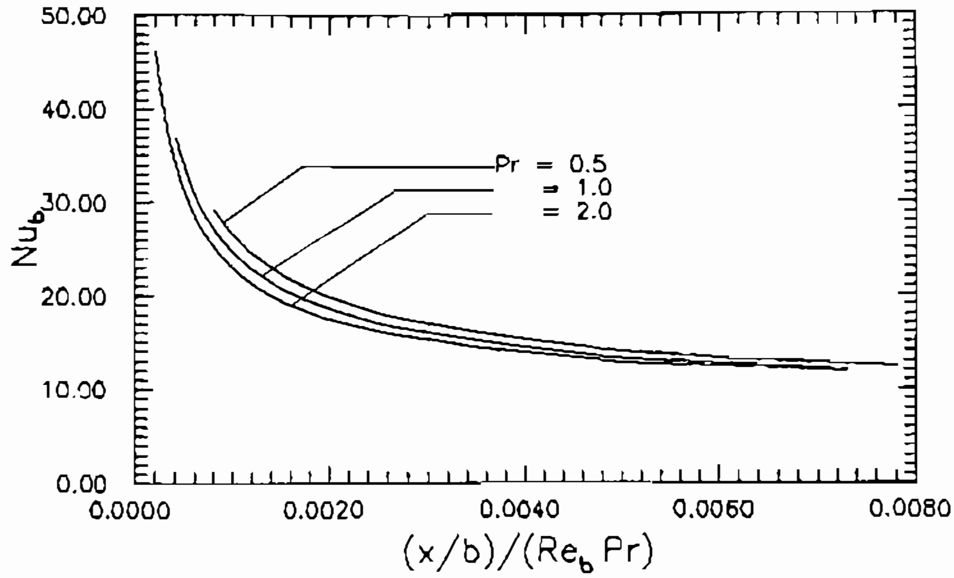

Fig. (7) Local Nusselt number in thermal entrance zone in case of constant wall temoeratures .

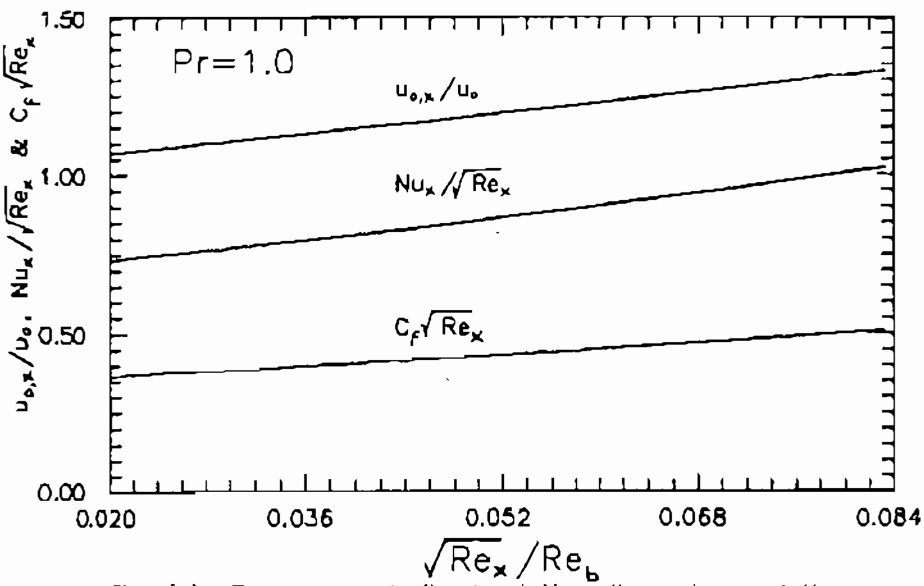

Fig. (8) The max. velocity, local Nusselt number and the coefficient of friction in entrance region in case of constant wall temperature .

$\sqrt{Re_x}/Re_b$	$Nu_x/\sqrt{Re_x}$ Pr=0.5	Pr=1.0	Pr=2.0
0.02000	0.58709	0.73634	0.92220
0.02083	0.59016	0.73951	0.92557
0.02174	0.59351	0.74297	0.92924
0.02273	0.59719	0.74677	0.93325
0.02381	0.60125	0.75095	0.93767
0.02500	0.60576	0.75557	0.94255
0.02632	0.61078	0.76072	0.94794
0.02778	0.61641	0.76647	0.95403
0.02941	0.62270	0.77296	0.96083
0.03125	0.63003	0.78032	0.96859
0.03333	0.63836	0.78874	0.97734
0.03571	0.64803	0.79848	0.98746
0.03846	0.65940	0.80986	0.99930
0.04167	0.67294	0.82334	1.01323
0.04545	0.68935	0.83955	1.02992
0.05000	0.70961	0.85942	1.05025
0.05556	0.73533	0.88432	1.07556
0.06250	0.77076	0.91643	1.10791
0.07143	.	0.95937	1.15070
0.08333		1.02035	1.20985
0.10000			1.29675
0.12500			1.49058

Table [1] Local Nusselt number in combined entrance region for Pr=0.5,1.0,2.0 .

[$Nu_x/\sqrt{Re_x}$ - $\sqrt{Re_x}/Re_b$]

Pr = 0.5		Pr = 1.0		Pr = 2.0	
x^*	Nu_b	x^*	Nu_b	x^*	Nu_b
0.00080	29.355	0.00040	36.817	0.00020	46.110
0.00087	28.332	0.00043	35.502	0.00022	44.434
0.00095	27.300	0.00047	34.176	0.00024	42.743
0.00103	26.273	0.00052	32.854	0.00026	41.058
0.00113	25.252	0.00057	31.540	0.00028	39.381
0.00125	24.230	0.00063	30.222	0.00031	37.702
0.00139	23.206	0.00069	28.902	0.00035	36.017
0.00154	22.189	0.00077	27.598	0.00039	34.342
0.00173	21.176	0.00086	26.282	0.00043	32.670
0.00195	20.161	0.00098	24.970	0.00049	30.993
0.00222	19.153	0.00111	23.664	0.00056	29.323
0.00255	18.147	0.00128	22.360	0.00064	27.653
0.00296	17.145	0.00148	21.058	0.00074	25.983
0.00347	16.149	0.00174	19.758	0.00087	24.316
0.00413	15.167	0.00207	18.472	0.00103	22.661
0.00500	14.193	0.00250	17.188	0.00125	21.005
0.00617	13.235	0.00309	15.916	0.00154	19.359
0.00781	12.332	0.00391	14.663	0.00195	17.727
		0.00510	13.432	0.00255	16.109
		0.00694	12.244	0.00347	14.519
				0.00500	12.968
				0.00780	11.925

Table [2] Local Nusselt number in combined entrance region for Pr=0.5,1.0,2.0 .

[Nu_b - x^*]