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CONCENTRATED FORCE ACTING ON AN ELASTIC INCLUSION IN A THICK-WALLED TUBE

by

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Abstract

A two-dimensional problem is investigated on the action of a concentrated force applied to the axis of a circular cylindrical, elastic inclusion embedded in an elastic thick-walled tube. This is a generalization of an indentation problem in an infinite space previously studied by Noble and Hussain [1] and revised by the author [2]. The problem is solved using a fast numerical approximation technique and numerical results are presented that allow to evaluate the angle of contact and to establish a comparison with the case of embedding in an infinite space.

1 Introduction.

Among the static problems of the Theory of Elasticity that have numerous applications in Engineering, the indentation problem is one of the most interesting. Due to its complexity, this problem is solved exactly only in very few special cases and numerically in general cases [3].

In their paper [1], Noble and Hussain reduce the problem of inclusion of an infinite circular cylinder in an infinite space to that of solving an airfoil integral equation, under the constraint that the elastic parameters of the media satisfy a certain relation. The same problem was treated by Omar and Hassan [2] who used a simpler technique to solve the dual series equations to which the problem was reduced in the general case. They showed, in particular, that sufficiently accurate results may be obtained from the first few iterations of their solution without need to transform to the integral equation.

The problem is solved following the same technique as in [2] and numerical results are given and discussed for the angle of contact between the inclusion and the tube. Comparison is established with the case of embedding in an infinite space. In particular, it is shown that the present results are the same as the corresponding ones in [2] when the shear modulus of the inclusion is much smaller than that of the outer medium.

2 Formulation of the problem

An infinite, isotropic, elastic circular thick-walled tube of radii a, b $(a < b)$ has an inclusion in the form of an infinite circular cylinder of radius a of another isotropic elastic material. A concentrated force F per unit length acts on the axis of the cylinder and perpendicular to it. Accordingly, a separation region establishes in the stressed medium, the bounds of which need to be determined. It is well-known that this problem reduces to the solution of a biharmonic equation for the stress function under proper conditions. This is further reduced to the solution of a pair of dual series equations involving the unknown angle of separation, the solution of which may be carried out numerically using an expansion in a small parameter, $\epsilon = a/b$, representing the ratio between the inner and outer radii of the tube. This allows to examine the case when the outer radius tends to infinity.

Let us introduce a set of cylindrical coordinates(r, θ, z) with z-axis coinciding with the axis of the inclusion, the force acting along the polar axis $\theta = 0$. In what follows, we briefly quote the fundamental equations to be used in the sequel. The same notations as in [1] will be used.

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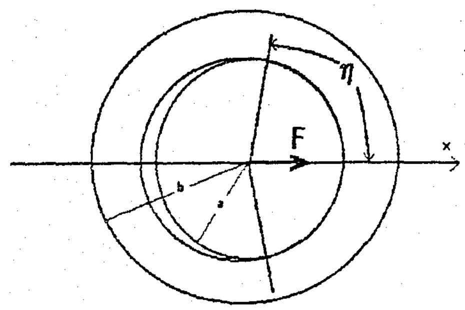


Fig. 1 Goematry of the problem.

(i) The stress components are expressed in terms of the stress function Φ as follows

$$\sigma_r = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}, \quad \sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right), \tag{1}$$

(ii) Strain–stress relations :

$$2G\epsilon_{rr} = (1-\nu)\sigma_r - \nu\sigma_\theta, \quad 2G\epsilon_{\theta\theta} = (1-\nu)\sigma_\theta - \nu\sigma_r, \quad 2G\epsilon_{r\theta} = \tau_{r\theta}, \tag{2}$$

where G and ν are the coefficient of rigidity and Poisson's ratio respectively.

(iii) Strain–displacement relations :

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), \quad 2\epsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}. \tag{3}$$

(iv) Boundary conditions :

Assuming a frictionless contact between the two bodies and a rigidly clamped outer surface of the tube, the boundary conditions are :

$$\begin{aligned}
\sigma_r(a,\theta) &= \sigma'_r(a,\theta), \quad 0 \le \theta \le \pi, & (4)\\
\tau_{r\theta}(a,\theta) &= \tau'_{r\theta}(a,\theta), \quad 0 \le \theta \le \pi, & (5)\\
u_r(a,\theta) &= u'_r(a,\theta), \quad 0 \le \theta \le \eta, & (6)\\
\sigma_r(a,\theta) &= 0, \quad \eta \le \theta \le \pi, & (7)\\
u_r(b,\theta) &= 0, \quad 0 \le \theta \le \pi, & (8)\\
u_\theta(b,\theta) &= 0, \quad 0 \le \theta \le \pi, & (9)
\end{aligned}$$

where the region of contact is $-\eta \le \theta \le \eta$, and the quantities referring to the inclusion are denoted by a "dash", while the undashed quantities are for the tube.

The boundary conditions must be completed with the condition of univaluedness of the displacement

$$u_\theta(r,0) = u_\theta(r,\pi) = 0, \quad 0 \le r \le b. \tag{10}$$

Also, the following global equilibrium condition should hold for both the inclusion and the tube :

$$F = -2\int_0^\pi (\sigma_r \cos\theta - \tau_{r\theta}\sin\theta)\, r\, d\theta. \tag{11}$$

From symmetry considerations, the stress functions Φ, Φ' may be shown to have the following expressions :

$$\begin{aligned}\Phi' &= \frac{aF}{4\pi(1-\nu')}\left[(1-2\nu')\rho\ \log\rho\ \cos\theta - 2(1-\nu')\rho\ \theta\sin\theta\right] \\ &\quad + A_0'\ \rho^2 + A_1'\ \rho^3\ \cos\theta + \sum_{n=2}^{\infty}\left[A_n'\ \rho^{n+2} + B_n'\ \rho^n\right]\ \cos n\theta, \end{aligned} \tag{12}$$

$$\begin{aligned}\Phi &= \frac{aF}{4\pi(1-\nu)}\left[(1-2\nu)\rho\log\rho\cos\theta - 2(1-\nu)\rho\theta\sin\theta\right] + A_0\rho^2 + \left[A_1\rho^3 + D_1\rho^{-1}\right]\cos\theta \\ &\quad + B_0\log\rho + \sum_{n=2}^{\infty}\left[A_n\ \rho^{n+2} + B_n\ \rho^n + C_n\ \rho^{-n+2} + D_n\ \rho^{-n}\right]\ \cos n\theta, \end{aligned} \tag{13}$$

where $\rho = r/a$ and $\{A_n'\}$, $\{B_n'\}$, $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, $\{D_n\}$ are coefficients to be determined.

The following expressions for the stresses and displacements in the two media are finally obtained:

(i) For the inclusion ($0 \le \rho \le \epsilon$)

$$\begin{aligned}\sigma_r' &= \frac{1}{2}E_0 + \frac{E_1\cos\theta}{4(1-\nu')}\left[(1-2\nu')\rho + (3-2\nu')\rho^{-1}\right] \\ &\quad -\frac{1}{2}\sum_{n=2}^{\infty}\left[(n-2)\rho^n - n\rho^{n-2}\right]E_n\cos n\theta \end{aligned} \tag{14}$$

$$\sigma_\theta' = \frac{1}{2}E_0 + \frac{E_1\cos\theta}{4(1-\nu')}\left[3\rho - \rho^{-1}\right] + \frac{1}{2}\sum_{n=2}^{\infty}\left[(n+2)\rho^n - n\rho^{n-2}\right]E_n\cos n\theta \tag{15}$$

$$\tau_{\theta r}' = \frac{E_1\sin\theta}{4(1-\nu')}\left[\rho - \rho^{-1}\right] + \frac{1}{2}\sum_{n=2}^{\infty} n\left[\rho^n - \rho^{n-2}\right]E_n\sin n\theta \tag{16}$$

$$\begin{aligned}2G'\frac{u_r'}{a} &= \frac{1}{2}(1-\nu')E_0\rho + \frac{2G'\delta}{a}\cos\theta + \frac{E_1\cos\theta}{8(1-\nu')}\left[(1-2\nu')(1-4\nu')\rho^2 + 2(3-4\nu')\log\rho\right] \\ &\quad -\frac{1}{2}\sum_{n=2}^{\infty}\left[\frac{n-2+4\nu'}{n+1}\ \rho^{n+1} - \frac{n}{n-1}\ \rho^{n-1}\right]E_n\cos n\theta \end{aligned} \tag{17}$$

$$\begin{aligned}2G'\frac{u_\theta'}{a} &= -\frac{2G'\delta}{a}\cos\theta + \frac{E_1\cos\theta}{8(1-\nu')}\left[(1-2\nu')(5-4\nu')\rho^2 - 1 - 2(3-4\nu')\log\rho\right] \\ &\quad +\frac{1}{2}\sum_{n=2}^{\infty}\left[\frac{n+4-4\nu'}{n+1}\ \rho^{n+1} - \frac{n}{n-1}\ \rho^{n-1}\right]E_n\sin n\theta \end{aligned} \tag{18}$$

(ii) For the tube ($\epsilon \le \rho \le 1$)

$$\begin{aligned}\sigma_r &= \frac{1}{2}E_0\frac{\epsilon^2 + (1-\nu)\rho^2}{(1-2\nu)+\epsilon^2} \\ &\quad + \frac{E_1\cos\theta}{4(1-\nu)}\left[\frac{(1-2\nu)(3-4\nu)-\epsilon^2}{(3-4\nu)+\epsilon^4}\rho^{-3} + (3-2\nu)\rho^{-1} + \frac{(1-2\nu)\epsilon^4+\epsilon^2}{(3-4\nu)+\epsilon^4}\rho\right] \\ &\quad - \sum_{n=2}^{\infty}\left[(n+1)(n-2)A_n\rho^n - n(n-1)B_n\rho^{n-2}\right. \\ &\qquad \left. + (n-1)(n+2)C_n\rho^{-n} + n(n-1)D_n\rho^{-n-2}\right]\cos n\theta. \end{aligned} \tag{19}$$

$$\begin{aligned}\sigma_\theta &= \frac{1}{2}E_0\frac{\epsilon^2 - (1-\nu)\rho^2}{(1-2\nu)-\epsilon^2} \\ &\quad - \frac{E_1\cos\theta}{4(1-\nu)}\left[\frac{(1-2\nu)(3-4\nu)-\epsilon^2}{(3-4\nu)+\epsilon^4}\rho^{-3} + (1-2\nu)\rho^{-1} - 3\ \frac{(1-2\nu)\epsilon^4+\epsilon^2}{(3-4\nu)+\epsilon^4}\rho\right] \end{aligned}$$

$$+\sum_{n=2}^{\infty}\Big[(n+1)(n+2)A_n\rho^{n}+n(n-1)B_n\rho^{n-2}$$
$$+(n-1)(n-2)C_n\rho^{-n}+n(n+1)D_n\rho^{-n-2}\Big]\cos n\theta, \qquad (20)$$

$$\tau_{\theta r} = \frac{E_1\sin\theta}{4(1-\nu)}\left[\frac{(1-2\nu)(3-4\nu)-\epsilon^2}{(3-4\nu)+\epsilon^4}\rho^{-3}-(1-2\nu)\rho^{-1}+\frac{(1-2\nu)\epsilon^4+\epsilon^2}{(3-4\nu)+\epsilon^4}\rho\right]$$
$$+\sum_{n=2}^{\infty}n\Big[(n+1)A_n\rho^{n}+(n-1)B_n\rho^{n-2}-(n-1)C_n\rho^{-n}-(n+1)D_n\rho^{-n-2}\Big]\sin n\theta, \qquad (21)$$

$$2G\frac{u_r}{a} = \frac{1}{2}\frac{1-2\nu}{1-2\nu+\epsilon^2}\left[\epsilon^2\rho-\rho^{-1}\right]E_0+\frac{E_1\cos\theta}{8(1-\nu)}\Big[2(3-4\nu)\log\epsilon\rho$$
$$+\frac{\Gamma_1\rho^{-2}+(1-4\nu)\Gamma_2\rho^2-\Gamma_3}{(3-4\nu)+\epsilon^4}\Big]-\sum_{n=2}^{\infty}\Big[(n-2+4\nu)A_n\rho^{n+1}$$
$$+nB_n\rho^{n-1}-(n+2-4\nu)C_n\rho^{-n+1}-nD_n\rho^{-n-1}\Big]\cos n\theta. \qquad (22)$$

$$2G\frac{u_\theta}{a} = \frac{E_1\sin\theta}{8(1-\nu)}\left[-2(3-4\nu)\log\epsilon\rho-2+\frac{\Gamma_1\rho^{-2}+(5-4\nu)\Gamma_2\rho^2-\Gamma_3}{(3-4\nu)+\epsilon^4}\right]$$
$$+\sum_{n=2}^{\infty}\big[(n+4-4\nu)A_n\rho^{n+1}+nB_n\rho^{n-1}+(n-4+4\nu)C_n\rho^{-n+1}$$
$$+nD_n\rho^{-n-1}\big]\sin n\theta. \qquad (23)$$

where δ is the rigid body displacement of the inclusion in the force direction and the coefficients are interrelated by the relations :

$$\Gamma_1 = \left[\epsilon^2-(1-2\nu)(3-4\nu)\right], \quad \Gamma_2 = \epsilon^2\left[(1-2\nu)\epsilon^2+1\right], \quad \Gamma_3 = (1-4\nu)-2(1-2\nu)\epsilon^2+\epsilon^4$$

$$E_1 = -\frac{F}{\pi a}, \qquad A_0' = \frac{1}{4}a^2E_0, \qquad A_1' = \frac{(1-2\nu')a^2}{8(1-\nu')}E_1,$$

$$A_0 = \frac{a^2\epsilon^2E_0}{4\left[(1-2\nu)-\epsilon^2\right]}, \qquad A_1 = \frac{a^2\epsilon^4E_1}{4\left[(3-4\nu)+\epsilon^4\right]},$$

$$B_0 = \frac{1-2\nu}{1-2\nu+\epsilon^2}\,a^2\,\frac{E_0}{2}, \quad D_1 = -\frac{a^2E_1}{8(1-\nu)}\,\frac{(1-2\nu)(3-4\nu)-\epsilon^4}{(3-4\nu)+\epsilon^4}.$$

and for $(n \geq 2)$:

$$A_n' = \frac{a^2E_n}{2(n+1)}, \quad B_n' = -\frac{a^2E_n}{2(n-1)}.$$

$$\begin{aligned}
(n+1)(n+2)A_n+n(n-1)B_n+(n-1)(n+2)C_n+n(n-1)D_n &= -E_n,\\
n(n+1)A_n+n(n-1)B_n-n(n-1)C_n-n(n-1)D_n &= 0,\\
(n-2+4\nu)\epsilon^{-n-1}A_n+n\epsilon^{1-n}B_n-(n+2-4\nu)\epsilon^{n-1}C_n-n\epsilon^{n-1}D_n &= 0,\\
(n+4-4\nu)\epsilon^{-n-1}A_n+n\epsilon^{1-n}B_n+(n-4+4\nu)\epsilon^{n-1}C_n+n\epsilon^{n-1}D_n &= 0.
\end{aligned}$$

The boundary conditions (6) and (7) give the following dual series equations in the unknowns E_0, $E_n(n \geq 2)$, δ and η :

$$\frac{1}{2}E_0+\sum_{n=1}^{\infty}E_n\cos n\theta = 0, \qquad \eta \leq \theta \leq \pi. \qquad (24)$$

Figure 12. The values of $\Delta\eta$ for $\nu = 0.0, \nu' = 0.5$

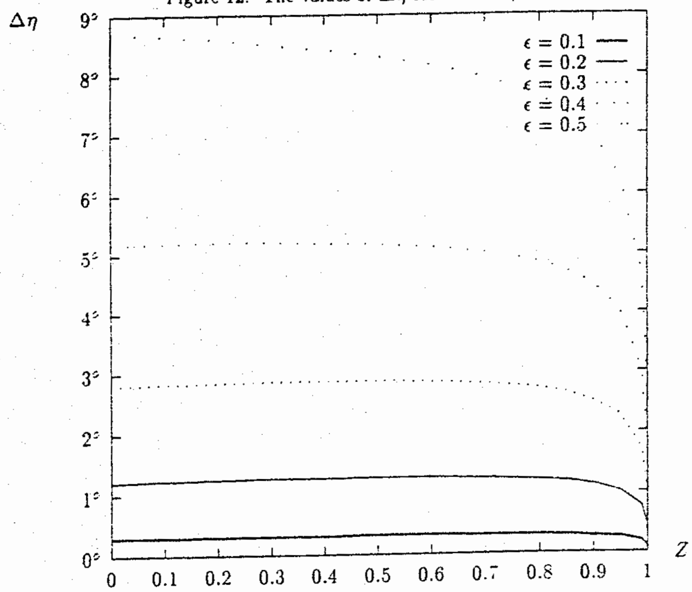
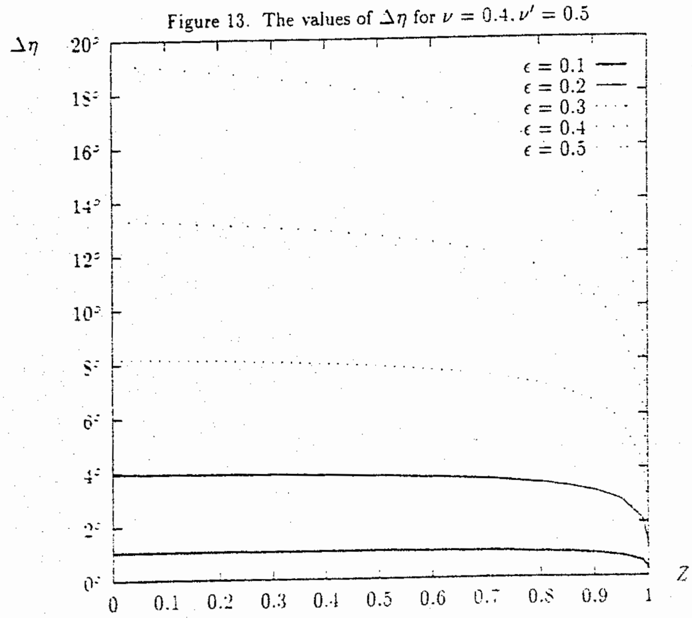


Figure 13. The values of $\Delta\eta$ for $\nu = 0.4, \nu' = 0.5$



$$\frac{1}{2}\, k_0\, E_0 + cE_1 \cos\theta + \sum_{n=2}^{\infty} \frac{\kappa_n}{n^2-1} E_n \cos n\theta = -\alpha\delta' \cos\theta, \qquad 0 \le \theta \le \eta, \tag{25}$$

where

$$k_0 = \frac{K_0}{2K_2}, \quad \kappa_n = \frac{K_1 - S_n}{2K_2}, \quad c = \frac{K_3}{2K_2}, \quad \delta' = \frac{G'\delta}{\alpha K_2}, \quad \alpha = \frac{G}{G'},$$

$$\begin{aligned} K_0 &= \alpha(1-2\nu') + \frac{(1-2\nu)(1-\epsilon^2)}{(1-2\nu)+\epsilon^2}, \\ K_1 &= \alpha(1-2\nu') - (1-2\nu), \\ K_2 &= \alpha(1-\nu') + (1-\nu), \\ K_3 &= \alpha L' + L, \end{aligned}$$

$$L = \frac{2\nu(3-\nu)\epsilon^4 - 4(1-2\nu)\epsilon^2 + (1-2\nu)3-4\nu) - 1 + 4\nu}{8(1-\nu)\left[(3-4\nu)+\epsilon^4\right]}, \quad L' = \frac{(1-2\nu')(1-4\nu')}{8(1-\nu')},$$

$$\begin{aligned} S_2 &= \frac{4(1-\nu)}{3-4\nu}\epsilon^2 \Bigg\{ 6\left[1+2(1-\nu)(1-2\nu)\right] - 6\left(1 + \frac{4\left[1+2(1-\nu)(1-2\nu)\right]^2}{3-4\nu}\right)\epsilon^2 \\ &\quad +2\left(1 + \frac{30\left[1+2(1-\nu)(1-2\nu)\right]}{3-4\nu} + \frac{48\left[1+2(1-\nu)(1-2\nu)\right]^3}{(3-4\nu)^2}\right)\epsilon^4 \\ &\quad +2\Bigg((3-4\nu) - \frac{18+16\left[1+2(1-\nu)(1-2\nu)\right]}{3-4\nu} - \frac{192\left[1+2(1-\nu)(1-2\nu)\right]^2}{(3-4\nu)^2} \\ &\quad - \frac{192\left[1+2(1-\nu)(1-2\nu)\right]^4}{(3-4\nu)^3}\Bigg)\epsilon^6 + O\left(\epsilon^8\right) \Bigg\}, \\ S_3 &= \frac{4(1-\nu)}{3-4\nu}\epsilon^4 \Bigg\{ 2\left[9+8(1-\nu)(1-2\nu)\right] - 24\epsilon^2 \\ &\quad + \left(9 - \frac{2\left[9+8(1-\nu)(1-2\nu)\right]^2}{3-4\nu}\right)\epsilon^4 + O\left(\epsilon^6\right) \Bigg\}. \\ S_4 &= \frac{4(1-\nu)}{3-4\nu}\epsilon^6 \left\{ 20\left[2+(1-\nu)(1-2\nu)\right] - 60\epsilon^2 + O\left(\epsilon^4\right) \right\}, \\ S_5 &= \frac{4(1-\nu)}{3-4\nu}\epsilon^8 \left\{ 3\left[25+8(1-\nu)(1-2\nu)\right] + O\left(\epsilon^2\right) \right\}. \\ S_n &= O\left(\epsilon^{2(n-1)}\right), n \ge 6, \end{aligned}$$

3 Solution of the dual series equations

To find the approximate solution of the dual series equations (24) and (25), we make use of the method suggested in [2]. Applying the operator $(D + D^{-1})$ on equation (25), where

$$D\, f \;=\; \frac{df}{d\theta}, \quad D^{-1}\, f \;=\; \int_0^\theta f(\theta)\, d\theta,$$

one gets

$$-\frac{1}{2}\, k_0 E_0\, \theta + \sum_{n=2}^{\infty} E_n \left(1 - \frac{\kappa_n}{n}\right) \sin n\theta = 0, \qquad 0 \le \theta \le \eta. \tag{26}$$

Equations (24) and (26) are sufficient to determine all the unknowns. Choose any integer $M \ge 1$, one can rewrite equation (26) in the truncated form

$$-\frac{1}{2} k_0 E_0 \theta + \sum_{n=1}^{\infty} E_n \sin n\theta = E_1 \sin\theta + \sum_{m=2}^{M+1} \frac{\kappa_m}{m} E_m \sin m\theta, \qquad 0 \le \theta \le \eta. \tag{27}$$

Thus the M-th order approximate solution of the dual series equations (24) and (27) woud be

$$E_n = a_{m1} E_1 + \sum_{m=2}^{M+1} \kappa_m \, a_{nm} \, E_m, \qquad n = 0, 1, 2, \cdots, \tag{28}$$

where $\{a_{ij}\}$ are the solution of the pair of dual series equations

$$\begin{aligned} -\frac{1}{2} k_0 a_{0m} \theta + \sum_{n=1}^{\infty} a_{nm} \sin n\theta &= \frac{\sin m\theta}{m}, \qquad 0 \le \theta \le \eta, \\ & \qquad\qquad m = 1, 2, \cdots, M-1, \\ \frac{1}{2} a_{0m} \theta + \sum_{n=1}^{\infty} a_{nm} \cos n\theta &= 0, \qquad \eta \le \theta \le \pi, \end{aligned} \tag{29}$$

which is given in [2].

Equations (28), for n running over the set of values $1, 2, \cdots, M+1$, form a set of $(M+1)$ homogeneous linear, algebraic equations in E_n $(1 \le n \le M+1)$. Since $E_1 \neq 0$, the determinant of the matrix of this system of equations must vanish, from which we can determine the angle η for the M-th order of approximation. One then calculates the values of the coefficients E_0 and $\{E_n\}$ $(n \ge 2)$ as in [2].

4 Numerical results and discussion

Some numerical calculations for the angle of contact η were carried out. Each one of the figures 2-7 shows the curves of the angle η against the physical parameter $z = \alpha \, / \, (\alpha + 1)$ for different values of the geometrical parameter ϵ and for a definite values of ν and ν'.

For the sake of comparison with the results of [2], we have plotted on figures 8-13 the difference $\Delta\eta$ between the actual angle and the corresponding one for the case $\epsilon = 0$ (case treated in [2]).

The results show that:

1. When $\nu = 0.5$, i.e. when the material of the inclusion is incompressible, the difference $\Delta\eta$ is almost constant as long as $\alpha \le 4$.

2. For $a/b \le 0.3$, the difference $\Delta\eta$ in general does not excced 10°, whatever the values of ν, ν' and α.

3. When $\alpha \to \infty$, i. e. when $G' \ll G$, $\Delta\eta \to 0$ whatever the values of ν, ν' and ϵ.

References

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Figure 2. The values of η for $\nu = 0.0, \nu' = 0.0$

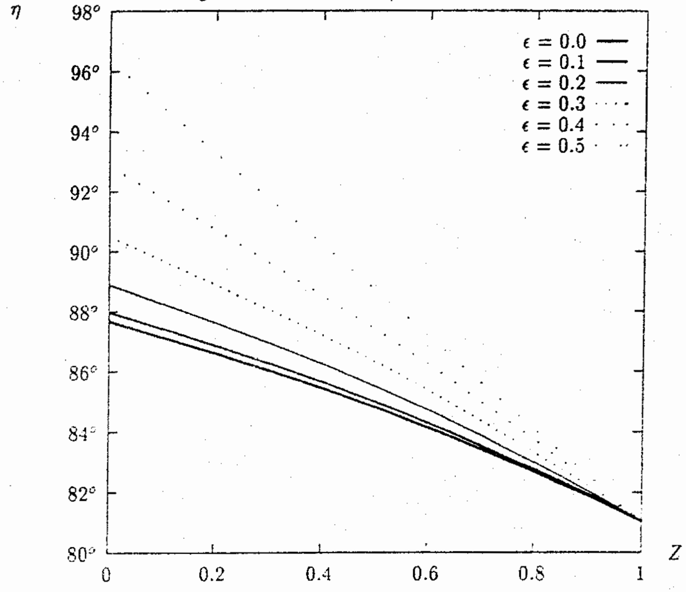


Figure 3. The values of η for $\nu = 0.1, \nu' = 0.1$

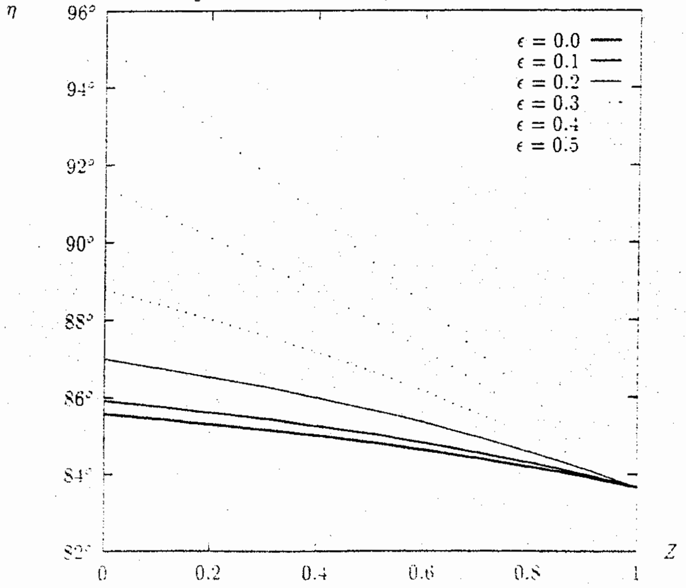


Figure 4. The values of η for $\nu = 0.2, \nu' = 0.2$

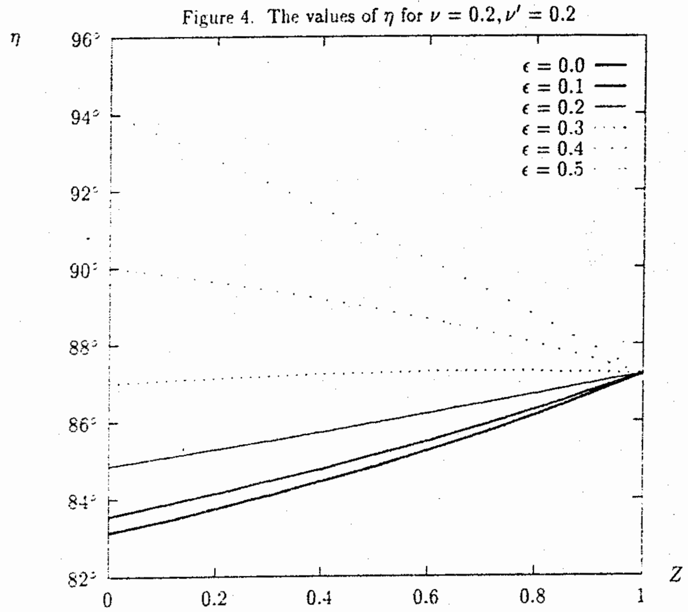


Figure 5. The values of η for $\nu = 0.0, \nu' = 0.3$

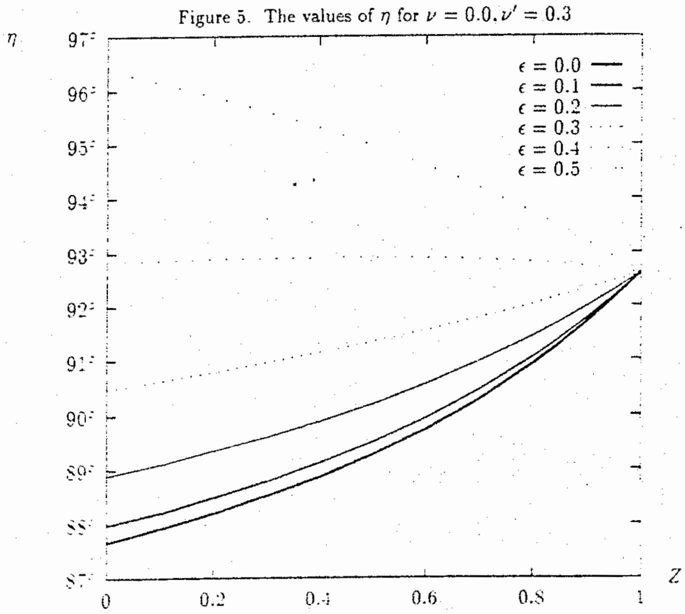


Figure 6. The values of η for $\nu = 0.0, \nu' = 0.5$

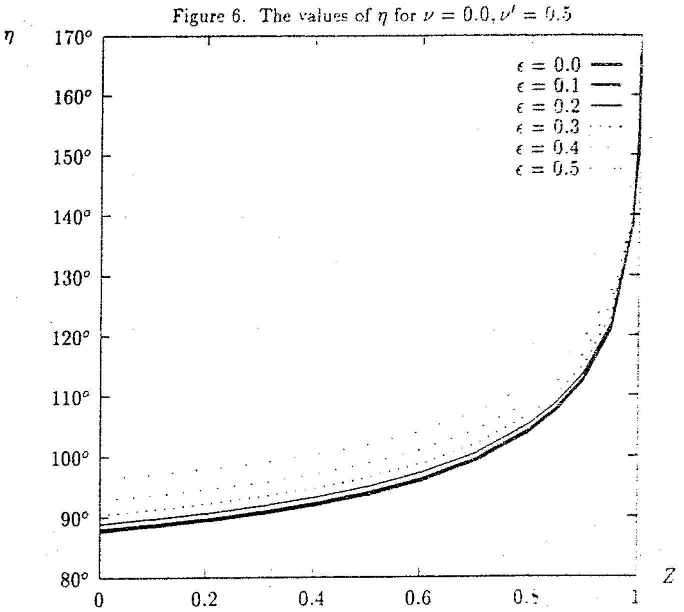


Figure 7. The values of η for $\nu = 0.4, \nu' = 0.5$

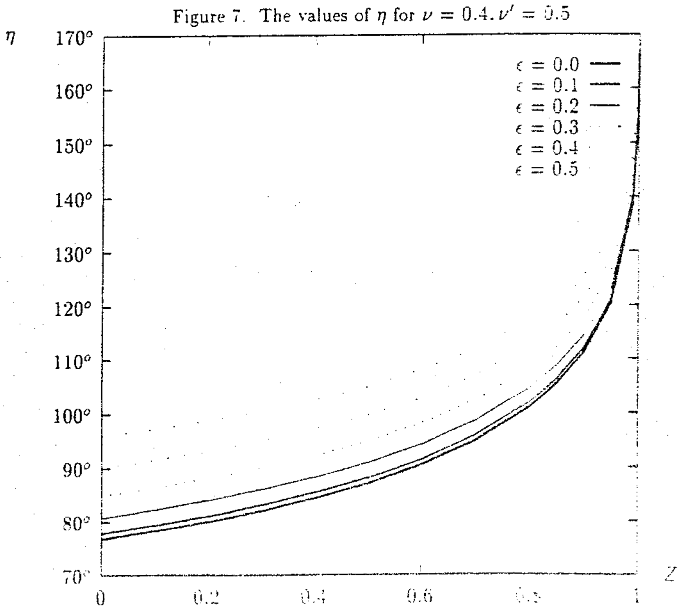


Figure 8. The values of $\Delta\eta$ for $\nu = 0.0, \nu' = 0.0$

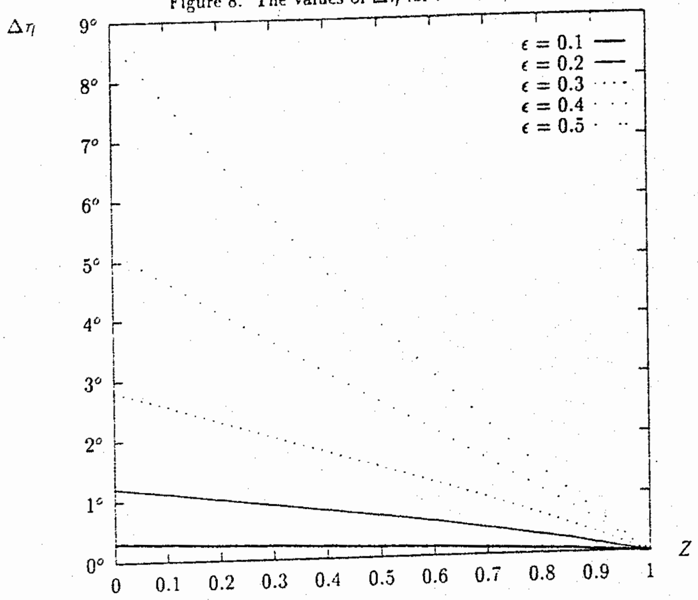


Figure 9. The values of $\Delta\eta$ for $\nu = 0.1, \nu' = 0.1$

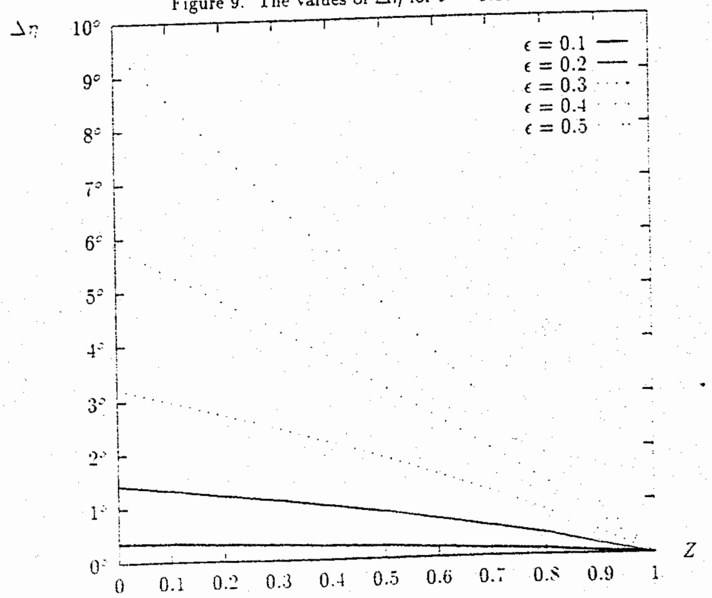


Figure 10. The values of $\Delta\eta$ for $\nu = 0.2, \nu' = 0.2$

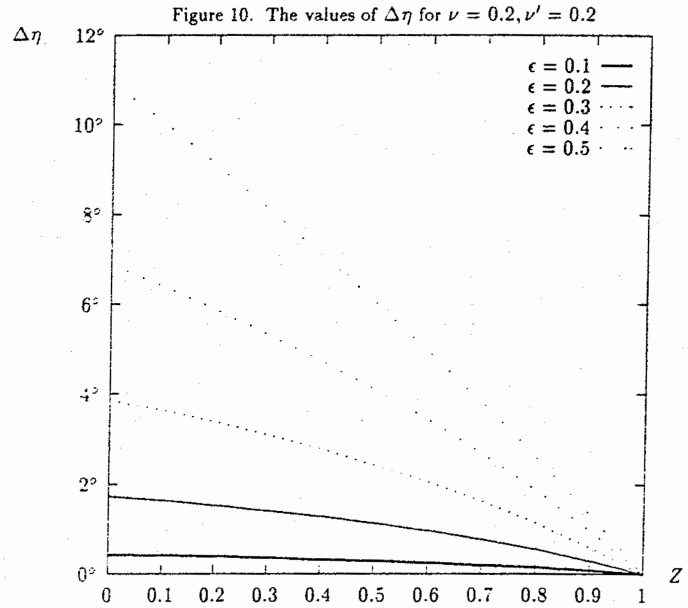
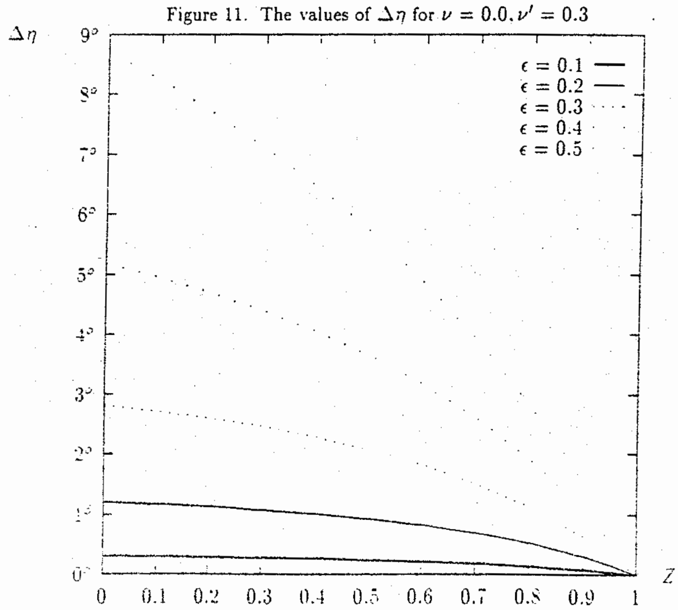


Figure 11. The values of $\Delta\eta$ for $\nu = 0.0, \nu' = 0.3$



تأثير قوة مركزة على جسم إسطوانى داخل أنبوبة

د . حسن أحمد زكى حسن

ندرس مسألة مستوية عن تأثير قوة مركزة تعمل فى محور إسطوانة دائرية مرنة داخلة فى أنبوبة سميكة . تعتبر هذه المسألة تعميما لمسألة سابقة فيها الجسم الخارجى لانهائى تمت دراستها بواسطة نوبل وصححت بواسطة المؤلف . تم حل المسألة باستخدام طريقة عددية سريعة التقارب وأعطيت النتائج العددية التى تتيح تعيين زاوية التلامس بين الإسطوانة والأمبوبة مع مقارنة بحالة الجسم الخارجى اللانهائى .