

SHREDDED WASTE TIRES AS A STABILIZING MATERIAL FOR UNPAVED ROADS

قطع الإطارات المتهاكة كمادة تثبيت للطرق غير المرصوفة

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خلاصة:

الغرض الأساسي من هذه الدراسة هو بحث إمكانية استخدام قطع الإطارات المتهاكة كمادة تسليح في إنشاء الطرق. وقد اختبر خليط من مادة الأساس للطريق (الرمال) وقطع الإطارات المتهاكة باستخدام كل من اختبار القص ثلاثي المحاور واختبار نسبة تحمل كاليفورنيا. وكانت العوامل المتغيرة في هذه التجارب هي نسب الخلط بالوزن وكذلك أحجام قطع الإطارات. وقد تم دراسة العوامل التالية لمعرفة تأثير قطع الإطارات على مقاومة القص ومقاومة الاختراق للخليط: الضغط المحصور، محتوى قطع الإطارات، حجم قطع الإطارات، كيفية وضع قطع الإطارات.

وقد خلصت الدراسة إلى أن قطع الإطارات المتهاكة تزيد من مقاومة القص ومقاومة الاختراق لطبقة الأساس. كذلك فإن محتوى قطع الإطارات المتهاكة وحجمها وكيفية إضافتها هي أهم العوامل المؤثرة على مقاومة القص والاختراق. وقد حدثت أفضل مقاومة قص عند نسبة قطع إطارات 6% وحجم قطع إطارات 5 × 5 مم، وأفضل مقاومة اختراق عند نسبة قطع إطارات 3% وحجم قطع إطارات 5 × 5 مم.

ABSTRACT:

The main objective of this study is to investigate the feasibility of using shredded waste tires as a reinforcing material in road construction. Triaxial and CBR tests were conducted on mixtures of base layer material (sand) and shredded waste tires. Sand-tire mixtures were prepared with different shred contents by weight and different shred sizes. These mixtures were subjected to identical laboratory tests to observe the influence of shredded waste tires on physical and engineering properties of sand-tire mixtures. The following parameters were studied to evaluate their influence on shear strength and resistance to penetration: confining pressure, shred content, shred size and order of adding shredded tires. Results indicated that generally, shredded waste tires increases the shear strength of sand and its resistance to penetration. Shred content, shred size and order of adding tire shreds are most significant characteristics of the mixes influencing shear strength and resistance to penetration. The higher improvement in shear strength is achieved with shred content equal to 6% and shredded tire size of 5×5mm in case of Triaxial test while in case of CBR test, the higher resistance penetration achieved with shred content of 3% and shred size of 5×5mm

KEYWORDS: Shredded waste tires – Unpaved road – Soil stabilization – Environmental

INTRODUCTION:

Many traditional methods were adopted to eliminate waste materials that are associated with bad effect on environment. Several studies deal with how to use the waste materials as an alternative or solutions in some applications of civil engineering. These studies are introduced to the fieldwork as a result of increasing the cost of elimination or recycling these materials. One of these materials is the waste tire that can be used as a construction material in road construction as presented below. The disposal of scrap tires has become a major environmental concern all over the

world. The disadvantages associated with a large store of waste tires draw the attention to develop new ways of reuse or recycle waste materials. The process of recycling tire system is complicated and expensive. Furthermore, the wishing of consumers to new product is more than the recycled products because new tires have a long life cycle than recycled tires. Recent research indicates that shredded waste tires do not show any likelihood of being a hazardous waste material or having adverse effects on ground water quality [1]. Waste tires can be used in the field and applications of earthwork. Shredded waste tires are now being used as subgrade reinforcement for constructing roads over soft soils, as aggregate in leach beds for septic systems, as an additive to asphalt, as a substitute for leachate collection stone in land fills and as sound barriers [2, 3]. A new design procedure for using shredded scrap tires as a lightweight fill material in highway construction was developed by Bosscher et al. [4]. They performed laboratory model test, field tests and numerical analyses to study embankments constructed using discarded shredded tires. The results of numerical analysis were showed that the FEM typically overpredicted the amount of displacement measured at the surface of the model test. Generally the results of this study supported the use of tire chips as an environmentally acceptable lightweight fill in highway applications of properly confined [4]. Tire shreds and soil-tire shred mixtures can be compacted using common compaction procedures. It is found that unit weight is primarily controlled by the amount of soil in the mixture, whereas compactive effort and molding water content appear to have little influence [2, 3, 5].

Direct shear tests were conducted on mixtures of outwash sand and tire shreds in a large-scale shear box. Edil and Bosscher [6] found that for dense outwash sand, adding 10% tire shreds by volume in a random arrangement resulted in greater strength than the pure sand. Fosse et al. [2] conducted direct shear tests on mixtures of dry sand and shredded waste tires. They reported that three significant factors affecting on shear strength were identified normal stress, shred content and sand matrix unit weight. The results of these tests ensure the adding shredded waste tires increased the shear strength of sand [2]. Also, the potential of tire rubber ash as a stabilizing agent for problematic soil material was investigated by Al-Homoud [7]. Soil-rubber ash mixtures were prepared with 0, 5, 7, 9, 11 and 20% by weight of both the natural and rubber treated soil. The following tests; Compaction, unconfined compressive strength, direct shear and free swell test were conducted on rubber ash admixtures to observe the influence of rubber on properties of the tested soils. Test results indicated that rubber ash is effective in stabilizing the soils and increase the shear strength of soil [7].

The main objective of the study described herein was to investigate the feasibility of using shredded waste tires as a means to enhance the stabilizing and performance of road construction. A series of Triaxial and CBR tests were conduct on mixture of sand and tire shreds to determine which factors influence their strength. The parameters that were studied included confining pressure, shreds content and shreds size, confining pressure with size and content of shred tires and order of shreds tire.

MATERIALS:

The materials used in the present study are basically sand and shredded tires. The sand used in this research was brought from Giza area. It was used to simulate base layer material in road construction. Primary tests were carried out to determine its physical and mechanical properties. Sand having particles size ranging from 0.1 to 2 mm, uniformity coefficient 2.85, and a specific gravity 2.63. The maximum dry density is 1.81 g/cm^3 and the optimum moisture content is 9.3%. The waste tires shreds used in this study were obtained from workshop for waste tires at Minia City. Shreds tire used was selected from sidewalk of whole tire because this part from the tire is free from wires. The tire shreds were divided into four groups based on size 5×5 , 10×10 , 20×20 and 30×30 mm. These sizes were used to study the effect of shreds tire size on the behaviour of randomly reinforced sand. Also, five different shreds tire content 1.5, 3, 4.5, 6 and 9 % by weight were used to study the effect of shreds content on the behaviour of reinforced sand.

EXPERIMENTAL PROGRAMME:

The testing programme consists of a series of Triaxial and CBR tests on pure sand and sand with inclusions of waste tire shreds. All tests were conducted at optimum moisture content and maximum dry unit weight. Modified compaction tests were carried out to determine the optimum moisture content and maximum dry density for pure sand and sand with inclusions of tire shreds. Fig.1 to Fig.4 shows the compaction curves for pure sand and sand with different sizes and contents of waste tire shred. From these figures it is clear that, an increase in shreds tire shreds content reduces maximum dry density and increases the optimum moisture content for all different size of shreds, these results agree with the result of Al-Homoud [7]. Typically behaviour can be observed with all different sizes of shreds tire used. The different parameters studied in CBR and Triaxial tests are confining pressure, shreds tire size, shreds tire content, order of shreds tire (random & sheets) and confining pressure with size and content.

Triaxial Tests

Triaxial tests with large size specimen having a diameter of 100 mm are conducted on pure sand and sand with shreds waste tire. A standard Triaxial apparatus was used, consisting of a motor-gear plated capable of moving vertically upwards or downwards at constant rate of loading 0.5 mm/min. The testing programme of this part of research involves the determining the difference between the behaviour of sand with and without shreds waste tire for identical samples at various test conditions (drained & undrained). Testing programme of Triaxial test is given in Table 1.

Table 1: Experimental Programme of Triaxial Tests

Test No.	Test Condition	Confining Pressure KN/m ²	Size of waste tire (mm)	Content of Waste tire (%)	Order of Waste tire
A	drained	100, 200, 300, 500	--	--	--
B	drained	300	5×5, 10×10, 20×20, 30×30	6	random
C	drained	300	5×5	1.5, 3, 4.5, 6, 9	random
D Effect of confining pressure with size	drained	100, 200, 300, 500	5×5	3	random
		100, 200, 300, 500	30×30		
E Effect of confining pressure with content	drained	100, 200, 300, 500	5×5	1.5	random
		100, 200, 300, 500		6	
F	drained	100, 200, 300, 500	Tire Sheets	1.21 2.43 3.72 4.98	One layer Two layers Three layers Four layers
G	undrained	100, 200, 300, 500	--	--	--
H	undrained	300	5×5, 10×10, 20×20, 30×30	6	random
I	undrained	100, 200, 300, 500	5×5 mm	1.5	random
		100, 200, 300, 500		6	

Tests were conducted on pure sand under four different confining pressures 100, 200, 300 and 500 kN/m². To study the effect of shreds tires size, tests were performed on samples mixed with different sizes of 5 × 5, 10 × 10, 20 × 20 and 30 × 30 mm under the same confining pressure of 300 kN/m² with shreds content of 6%. Also, tests were carried out on samples having shreds of 5 × 5 mm in size under constant confining pressure with different shreds contents of 1.5, 3, 4.5, 6 and 9% randomly mixed with sand. All tests are conducted at optimum moisture content and maximum dry density as mentioned before. To investigate the effect order of shreds tire, shreds tire is cut to into

disc shape element of approximately the same diameter as the Triaxial specimen and placed horizontally into the sand specimen.

CBR Tests:

Base layer material (sand) with or without shreds tire randomly mixed was placed in CBR mould in five layers and subjected to 56 blows per layer using 4.5 Kg weight with a drop of 450 mm, according to ASTM-D1557-70. A surcharge of 4.5Kg weight in the form of annular ring was placed in the top of the soil surface and 500-mm diameter standard CBR plunger was pushed through the annular steel ring into the soil at a constant rate of 1mm/min. In the soaked samples, the specimen is soaked for a period 96 hours with a surcharge weight 4.5Kg. The CBR values were determined at 2.5 and 5 mm for both top and bottom of the specimen. The testing programme of CBR tests involves the determining the difference between the resistance of sand penetration with and without inclusions of shreds tire for identical samples. All tests are carried out in both unsoaked and soaked samples having shreds tire with different sizes of 5 × 5, 10×10, 20×20 and 30×30 mm with different ratios of 1.5, 3, 4.5, 6 and 9%.

ANALYSIS OF TRIAXIAL TEST RESULTS:

The relationship between the deviator stress ($\sigma_1 - \sigma_3$) and axial strain ϵ % for pure sand are shown in Fig.5. From this figure, it can be observed that the effect of increasing the confining pressure is associated with an increase in deviator stress at any strain. As illustrated in testing programme, two test series B and C have been performed on sand with randomly distributed shredded tire elements. To study the effect of size and content of shredded waste tires four sizes and five contents have been used as mentioned before. It may be noted that by mixing the sand with a large number of shreds waste tire elements in a random inanner, a new material is formed having properties different from these of the sand without additives.

Fig.6 shows the relationship between deviator stress and axial strain for pure sand and sand with different sizes (5×5, 10×10, 20×20 and 30×30mm) of shreds tire tested under constant confining pressure of 300 KN/m² and shreds tire content of 6%. It can be observed that the presence of shredded waste tires improves the stress-strain properties for all different sizes of shredded waste tire. In all cases the maximum deviator stress occurs at a higher axial strain compared with the case of pure sand. Also, from this figure, one can notice that the maximum improvement in the deviator stress achieved with size 5×5mm compared to other sizes. Beyond this size, no much difference between the values of improvement in the deviator stress for all different sizes of shredded waste tires (10×10, 20×20 and 30×30 mm).

The percentage improvement in maximum deviator stress as a result of additive inclusions of shreds tire to the specimen can be compacted from the following relation:

$$PI = \frac{\text{Max. deviator stress for sand with tire} - \text{Max. deviator stress for pure sand}}{\text{Max. deviator stress for pure sand}}$$

The relation between percentage improvement in max. deviator stress against size of shredded waste tires are plotted in Fig.7. From this figure it is clear that, the best performance can be achieved in case of using size 5×5 mm with a value of 35%. Also, it can be noted that no much difference between the percentage improvement of maximum deviator stress for other sizes. Fig.8 shows the relation between deviator stress and axial strain for pure sand and sand with different contents (1.5, 3, 4.5, 6 and 9% by weight) of shreds tire tested under constant confining pressure of 300 KN/m² and constant size (5×5 mm). From this figure it can be seen that for all different contents of shreds tire, the deviator stress is higher for sand with shreds tires than pure sand especially at values of high strains. The relation between percentage improvement in max deviator stress against shreds tire content are shown in Fig.9. A greater improvement in stress-strain behaviour is achieved by increasing the shredded waste tire content until 6 % and then decreases.

Two series of tests **D** and **E** were conducted to study the effect of confining pressure with size and content of shreds tire as mentioned in testing programme. Series **D** was conducted on two different sizes of shreds waste tire content (5×5 mm and 30×30 mm) and constant shreds tire content of 3 % for all different confining pressure 100, 200, 300 and 500 KN/m². Series **E**, was performed on two different contents 1.5 and 6 % constant at constant shreds tire size of 5×5 mm for all different confining pressures. The relations between percentage improvement in max deviator stress against confining pressure are plotted in Fig 10 for case of shreds size effect and Fig 11 for case of shreds tire content. From these figures it is clear that higher percentage improvement in maximum deviator stress is obtained at lower confining pressure. Also, it can be observed that the increasing in percentage improvement in maximum deviator stress are achieved at size (5×5 mm) and shreds tire content of 6 %.

Series **F** were conducted on sand including horizontal circular tire sheets of approximately the same diameter of the triaxial test specimen as illustrated in testing schedule. Fig.12 shows the stress-strain relationship at constant confining pressure 300 KN/m² for pure sand and sand with one, two, three and four tire sheets located at equal distances between each layer which correspond to waste tire contents of 1.21, 2.43, 3.72 and 4.98 % respectively. Again presence of reinforcement in the sample increases the deviator stress especially at high values of strains. Also, it can be observed that the maximum deviator stress occurs at higher strain values compared with pure sand.

For a better illustration, the relation between the percentage improvement of maximum deviator stress against tire sheets layers numbers are plotted in Fig.13. From this figure it is clear that, the percentage improvement in maximum deviator stress increases with increase the number of tire sheets. For comparison between the behaviour of randomly reinforced and reinforced with parallel tire sheets, the relation between maximum deviator stress against percentage waste tire content for both tire sheets and randomly shreds tire with size (5×5 mm) is plotted in Fig.14. Noting that the tire sheet reinforcement tests were conducted approximately at the same waste tire content in case of random reinforcement. It can be seen from this figure that tire sheets gave better improvement compared to random reinforcement at the same waste tire content.

Triaxial Results under Undrained Tests:

Unlike drained Triaxial shear test, no volume change is allowed during axial loading in an undrained Triaxial shear test [8]. In our study the behaviour of sand under undrained conditions could be considered as a result of two reasons. Firstly, in case of earthquake condition, under the transient loading of an earthquake shock there is not enough time for even sand to dissipate pore water pressure by drainage and the undrained condition applies. The second reason, when the cross sectional of road covers with water (Saturation State) due to floods or rainfall. The sequence movements for wheels of vehicles or trucks over roads surface profile are considered a cyclic (dynamic) loading. So, under its loading movement, road layer soils has not enough time to dissipate pore water pressure by drainage. Therefore, the soil under this loading (dynamic) would behave undrained condition. Based on previous reasons, the stress-strain behaviour of base layer (sand) material in case of reinforced and unreinforced under undrained condition should be considered. The relationship between the effective deviator stress ($\sigma_1 - \sigma_3$) and axial strain ϵ % for pure sand under undrained condition for all different confining pressures are shown in Fig.15. From this figure, one can observe that, increasing confining pressure is associated with an increase in effective deviator stress at any strain value. Based on previous results of randomly reinforced sand with shredded waste tires under drained condition, the maximum improvement in stress-strain behaviour was obtained in case of 6 % tire content with tire size (5×5 mm) and the minimum improvement was obtained at 1.5 % tire content. According to schedule of testing programme, series **G** was conducted on minimum and maximum contents (1.5 and 6 %) respectively with size (5×5 mm) for all different confining pressure to study the effect of shreds waste tire content under undrained condition.

Also, to study the effect of shredded waste tire size under undrained condition. Series H was conducted on the same previous sizes that are used in drained condition (5×5, 10×10, 20×20 and 30×30 mm) with constant tire content 6 % and constant confining pressure 300 KN/m². The relations between percentage improvement in max. deviator stress against confining pressure are plotted in Fig. 16. From this figure it is clear that, the percentage improvement in maximum deviator stress decreases with increase the confining pressure for both drained and undrained tests. The percentage improvement in undrained condition is higher than drained condition for all different confining pressure. It can also notice that both samples tested under drained and undrained condition gave the same value of improvement. The trend of improvement in undrained tests is much pronounced than that in case of drained tests.

To illustrate the effect size of shredded waste tire on randomly reinforced sand under undrained condition, the relations between total deviator stress and axial strain are plotted in Fig. 17. As observed before in stress-strain behaviour under drained condition, the maximum improvement was obtained with size (5×5 mm) and no much difference between other sizes. The relation between max. deviator stress against size of shredded waste tire for both drained and undrained condition is shown in Fig. 18. It can be seen that there is no much difference between the shape of both curves for both drained and undrained conditions. The stress-strain behaviour of randomly reinforced in undrained is more significant than drained condition.

CBR Test Results

As mentioned before, the main purpose of this research is to assist the feasibility of using shredded waste tires as a reinforcement material in road construction. CBR tests were conducted on base layer (sand) material without and with different shredded waste tires for the same parameters mentioned for Triaxial tests. The main objective of CBR tests is to determine the penetration resistance for pure sand and sand with different inclusions of shredded waste tires and to investigate the effect of additives shreds waste tire to sand on the CBR value. As mentioned before four sizes (5×5, 10×10, 20×20 and 30×30 mm) and five contents (1.5, 3, 4.5, 6 and 9%) are used in this research. All specimens were tested in case of unsoaked and soaked condition. To study the effect of size of shreds tire, the relation between size of shredded waste tire against CBR value and percentage improvement in CBR are plotted in Fig. 19 and Fig. 20. From these figures it can be observed that, the higher improvement in CBR value are achieved at shreds size (5×5mm) and no much difference in (%) improvement in CBR values for other sizes for both soaked and unsoaked condition. Also, one can be noted that, the effect of shreds tire size agrees with results of Triaxial test. Fig. 21 shows the relation between CBR values against shredded waste tires content at constant size of shreds tire size (5×5mm). It is clear that the CBR values increase with the increase of shreds tire content up to 3 % content in both soaked and unsoaked specimens. After 3% content the CBR value decreased with the increasing of shreds tire content. For a better illustration, the percentage improvement in CBR values against shreds content is plotted in Fig. 22. It can be noted that the percentage improvement in CBR value in case of 3% content reaches the value of 36% and after this content the percentage improvement in CBR value decreased to reach approximately zero in case of 9% content. These results are consistent with other results of many researchers [9,10,11], although they used different types of reinforcing (stabilizing) material such as geogrid, geotextile, polymeric mesh elements. One can be say that, the higher improvement in CBR value for sand with inclusions of shredded waste tires was obtained at 3% content in case of size (5×5mm). Therefore 3% is consider the best content of shredded waste tires additives for increasing the CBR value for both soaked and unsoaked condition. These result can be explained the phenomena of decreasing the (%) improvement in CBR value after 3% content is related to the higher compressibility of sand-tire mixture beyond this percentage. Therefore, the resistance of penetration decreased with the increasing the compressibility of material.

CONCLUSIONS

Based on test results, the following conclusions are drawn:

1. An increase in shreds tire content reduces the maximum compaction dry density and increases the optimum moisture content for all different shreds size.
2. Increasing the confining pressure is associated with an increase the deviator stress at any strain in both drained and undrained condition.
3. The presence of shredded waste tires in sand improves the stress-strain properties for all different sizes and contents of shreds waste tire over that pure sand. The maximum deviator stress of randomly reinforced sand occurs at a higher axial strain compared to sand alone.
4. The maximum improvement in the deviator stress is achieved with size (5×5mm) and beyond this size, no much difference between the values of improvement on deviator stress for all different sizes. It may be said that, shreds tire size has no significant effect on shear strength expect size (5×5mm) in both drained and undrained condition.
5. A greater improvement in the stress-strain behaviour can be achieved by increasing the shreds tire content till 6 % after this content, the improvement in the stress-strain behaviour has no effect.
6. The higher improvement in stress-strain behaviour is obtained when the inclusions of waste tires (tire sheets) are placed horizontal directions compared to randomly reinforced sand with shreds tire at the same waste tire content.
7. The percentage improvement in maximum deviator stress decrease with the increase of the confining pressure for both drained and undrained tests.
8. The effect of randomly reinforced sample tested in undrained condition is more significant than drained condition for all different confining pressure expect higher confining pressure (500 KN/m²).
9. CBR values increases with the increase of shreds tire content up to 3 % content. After this content the increasing of CBR value decreases with the increase of shreds tire content in both soaked and unsoaked specimens.
10. The higher improvement in CBR value are achieved with tire size (5*5 mm) and no much difference in (%) improvement in CBR values for another sizes in both soaked and unsoaked tests. This result agrees with results of Triaxial test.
11. Shreds tire with 3% content and shreds tire size (5*5 mm) are consider the best output in the improvement of CBR value.

ACKNOWLEDGMENTS:

The data of this paper was taken from Eng. A.A.Ali who works in his Ph.D thesis under the supervised of the authors and he still works in his Ph.D thesis till the date of publish this paper. The authors would like to express their thanks to him.

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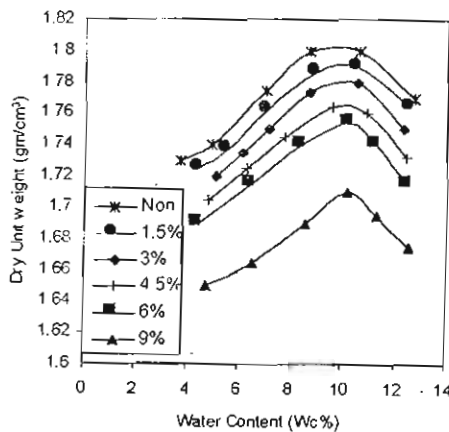


Fig.1. Modified Compaction Curve for Sand-Tire Mixture with shreds size (5*5mm) at different tire contents.

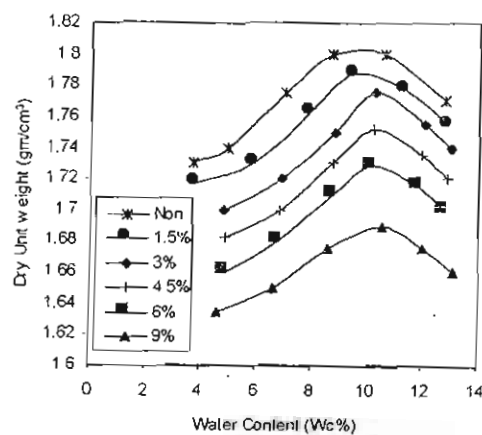


Fig.2. Modified Compaction Curve for Sand-Tire Mixture with shreds size (10*10mm) at different tire contents.

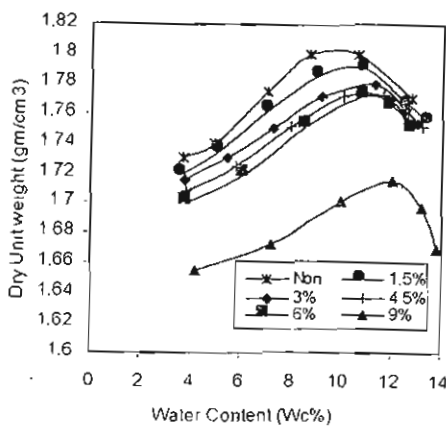


Fig.3. Modified Compaction Curve for Sand-Tire Mixture with shreds size (20*20mm) at different tire contents.

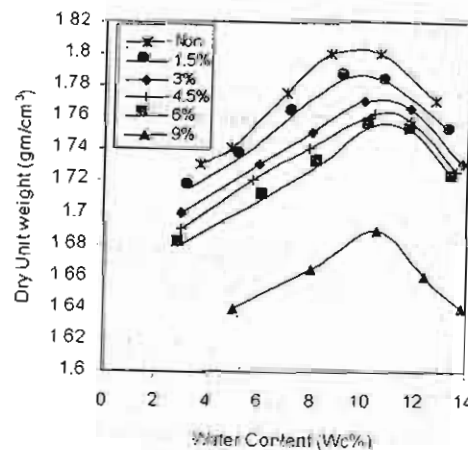


Fig.4. Modified Compaction Curve for Sand-Tire Mixture with shreds size (30*30mm) at different tire contents.

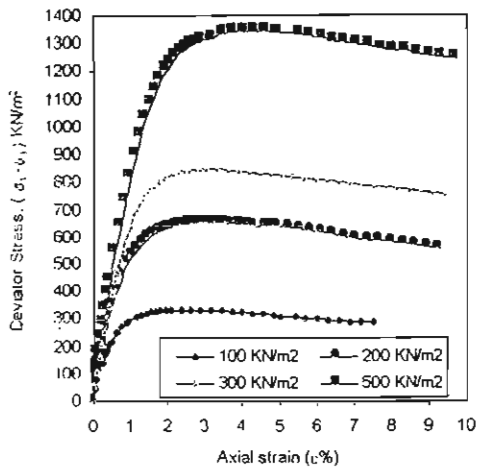


Fig. 5. Relation between deviator stress and axial strain for pure sand at different confining pressure.

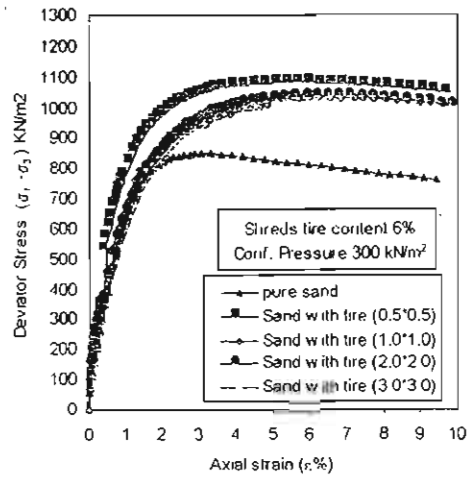


Fig. 6. Relation between deviator stress and axial strain for pure sand and sand with different sizes of shred tires at constant confining pressure 300 kN/m² and 6% shreds content

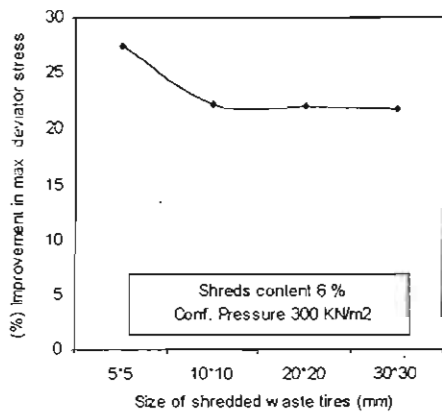


Fig. 7. Effect of tire sizes on the (%) improvement of max. Deviator stress at constant confining pressure 300 kN/m² and 6% tires content.

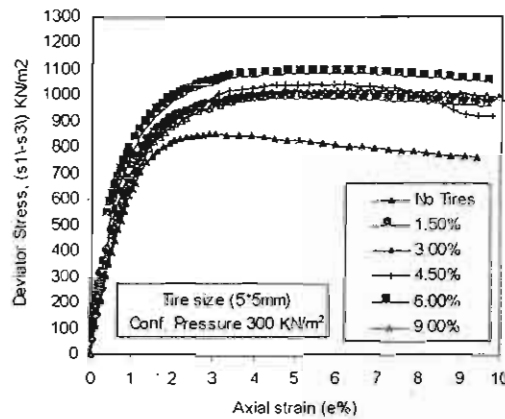


Fig. 8. Relation between deviator stress and axial strain for pure sand and sand with different percentages of shred tires content for size (0.5*0.5) at constant confining pressure 300 kN/m².

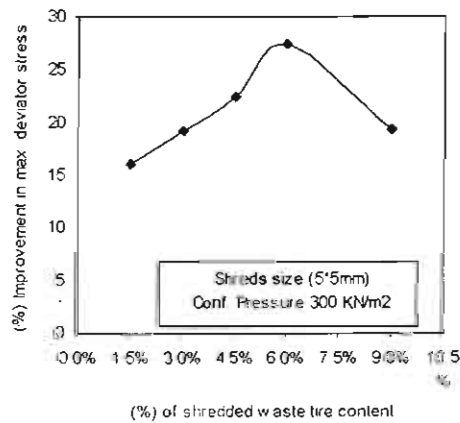


Fig. 9. Effect of shreds tire content on the improvement of max. deviator stress at confining pressure 300 kN/m² and tire size (0.5*0.5)

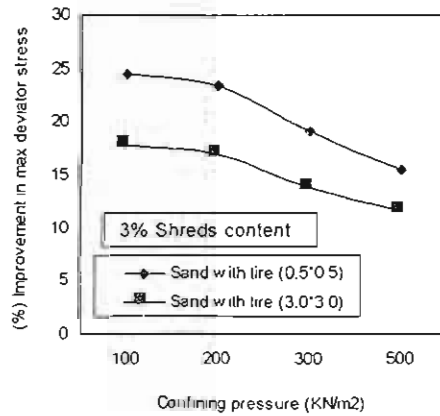


Fig. 10. Effect of confining pressure with size of shredded waste tires.

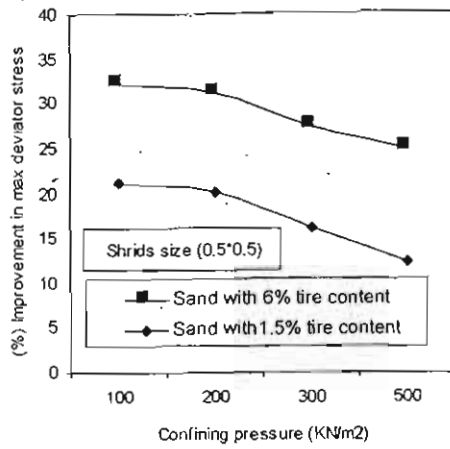


Fig. 11. Effect of confining pressure with shreds waste tire content.

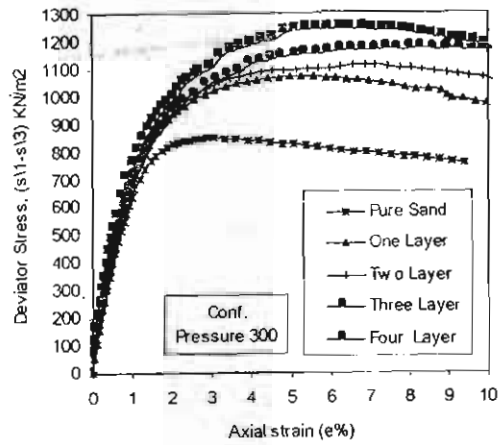


Fig. 12. Stress-strain relationship for pure sand and sand with different sheets tire at constant confining pressure 300 KN/m².

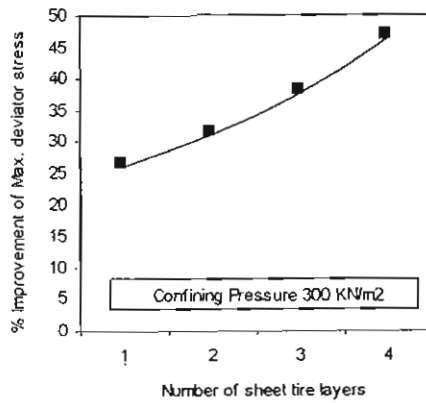


Fig. 13. Variation of % improvement of max. deviator stress with number of tire layers.

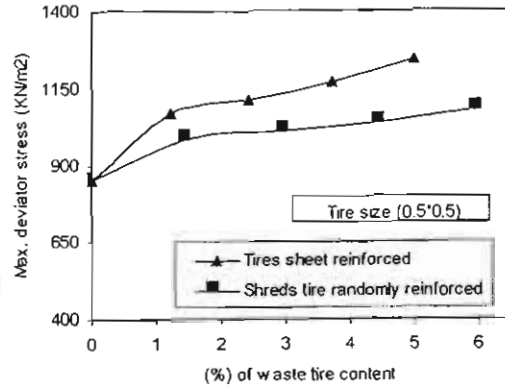


Fig. 14. Variation between max. deviator stress and waste tire content in both sheet layer and randomly reinforced at 300 KN/m².

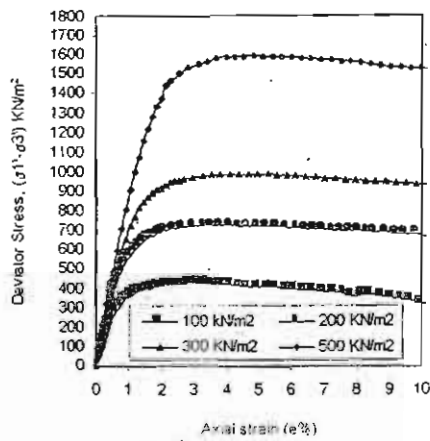


Fig. 15. Stress-strain relationship for pure sand in maintained condition.

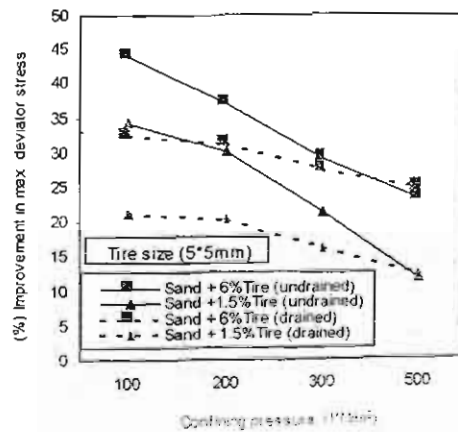


Fig. 16. Effect of confining pressure with tire contents in both drained and undrained condition at constant tire size (5*5mm).

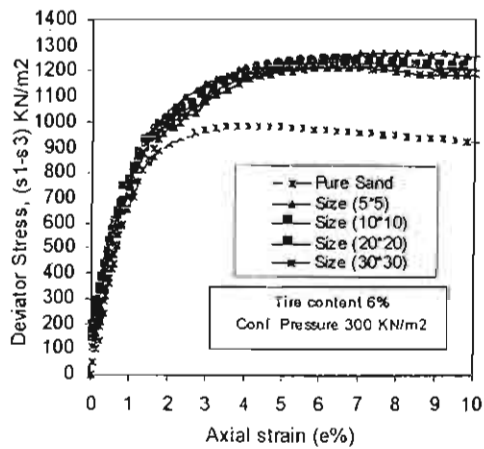


Fig. 17. Stress-strain relationship for pure sand and sand with different sizes in case of undrained condition at constant 6% content.

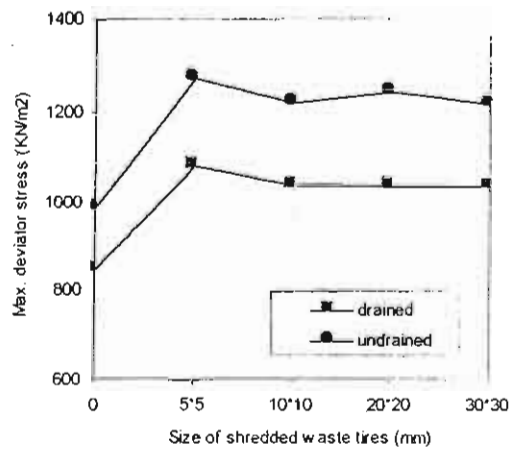


Fig. 18. Effect of shreds size on the max. deviator stress at constant conf. pressure 300 KN/m².

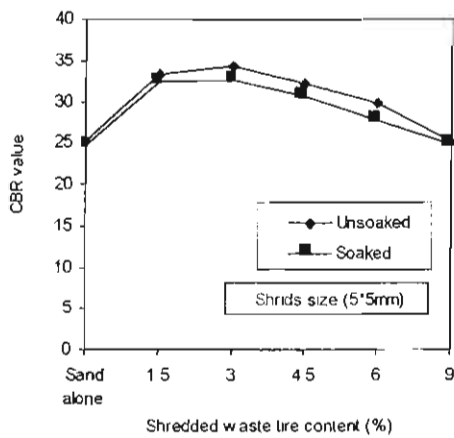


Fig. 19. Effect of shred size on CBR values in both soaked and unsoaked.

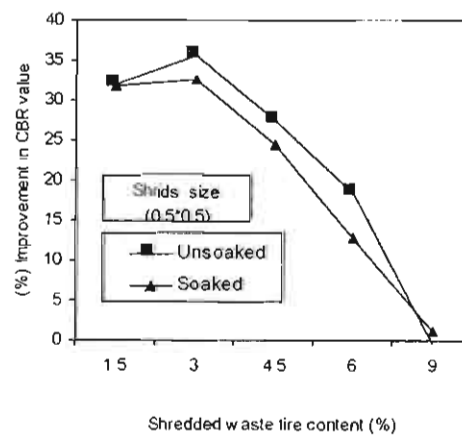


Fig. 20. Effect of shreds size on the (%) improvement of CBR values.

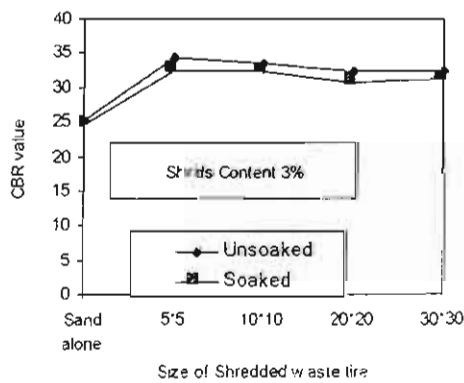


Fig. 21. Effect of Shredded Waste Tire Content in the CBR values.

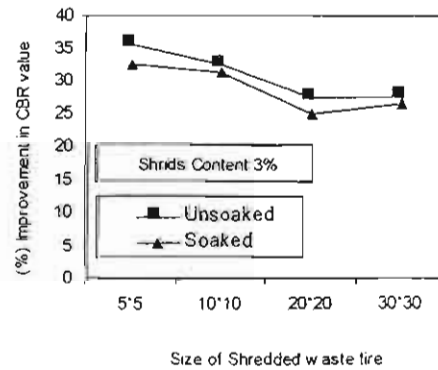


Fig. 22. Effect of shred contents on the (%) improvement of CBR values.

TRANSIENT GROUNDWATER FLOW IN HETEROGENEOUS GEOLOGICAL FORMATIONS

السريان الانتقالي للمياه الجوفية فى التكوينات الجيولوجية غير المتجانسة

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خلاصة:

في هذا البحث تم عرض محاكاة عددية للسريان غير المستقر للمياه الجوفية في طبقات متجانسة و أخرى غير متجانسة. تم توليد البناء غير المتجانس من الأنماط الجيولوجية باستخدام نموذج سلسلة ماركوف المزدوج. تم حل المعادلات الحاكمة للسريان باستخدام طريقة الفروق المحدودة للسريان غير المستقر في المياه الجوفية في المستوى الأفقى و بخصائص غير متجانسة للطبقات.

أوضحت النتائج أن التعريف في مقدار الانحدار الهيدروليكي العام مع عدم التجانس في الطبقات يؤدي إلى تعريفه في مقدار واتجاه الانحدار الهيدروليكي المحلي. كما أوضحت النتائج أن زيادة معامل التخزين يؤدي إلى تعميم في رد فعل الطبقة الحاملة للمياه في صورة الضاغط الهيدروليكي و سرعة دارسي. و هذا التعميم أكثر في حالة الضاغط الهيدروليكي عنه في سرعة دارسي.

أظهرت النتائج أيضا أنه في حالة الطبقات غير المتجانسة و التي درست في هذا البحث أن الضاغط الهيدروليكي و سرعة دارسي الجانبية كانت في طور واحد بينما سرعة دارسي في الاتجاه الطولي كانت خارج الطور.

ABSTRACT

Numerical simulations of unsteady groundwater flow in homogeneous and artificially generated heterogeneous geological formations have been presented. The heterogeneous structure of the geological patterns has been generated using the coupled chain Markov model developed by Elfeki and Dekking [2001]. Solution of the governing equations is achieved through the application of a finite difference approach to the partial differential equation of unsteady groundwater flow in horizontal plane with heterogeneous properties. It has been shown that global gradient (regional gradient) magnitude variability coupled with aquifer heterogeneity generates local directional and magnitude gradient variabilities. Increasing of the storage coefficient leads to smoothing of the aquifer response in terms of hydraulic head and Darcy's velocity. However, the smoothing effect is more pronounced in the hydraulic heads when compared with Darcy's velocity. In heterogeneous aquifer presented in this study, the aquifer response in terms of hydraulic head field and the lateral Darcy's velocity are in phase with the input time series, however the longitudinal Darcy's velocity is out of phase

1. INTRODUCTION

Transient flow conditions have strong influence on contaminant spreading in aquifers.

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This behavior has been supported by many field observations [e.g. Gelhar, 1993]. Significant progress of steady groundwater flows in stationary Gaussian and non-Gaussian random fields have been achieved [e.g. Smith and Freeze, 1979 and Ababou et al., 1989]. Many researchers show still vivid interest to describe the hydrodynamics of flow in heterogeneous fields under transient conditions. Only a limited number of studies are devoted to this area. Just recently, in the hydrogeological community, a considerable attention is made on evaluating the effects of transient conditions in heterogeneous media.

Two main transient conditions are causing the spreading: the gradient magnitude variability and the gradient direction variability. In the current research, a focus is made on the influence of gradient magnitude variability, which can be described as a multiple scale time series, combined with aquifer heterogeneity on flow characteristics.

The goal of this research is to investigate the hydrodynamics of flow under transient conditions in heterogeneous aquifer. Unsteady groundwater flow model in a heterogeneous confined aquifer has been developed. The model is based on a finite difference numerical scheme in terms of potentials. The model is used to study the influence of transient conditions (gradient magnitude variability) on groundwater flow behavior at multiple time scales. In this model, the influence of water level fluctuations in a river, that is feeding an aquifer (see Figure 1), on the hydraulic head and local Darcy's velocity fluctuations are considered. Simulations have been performed by the developed model under two different input signals in homogeneous and heterogeneous media with a large scale spatial variability. The first signal is a sudden drop in the water level in the river side and the second is a time series in a form of two components a random signal superimposed over a cosine wave.

2. GOVERNING EQUATIONS OF UNSTEADY GROUNDWATER FLOW PROBLEM

The governing equation, in the absence of source and sink terms, of unsteady two-dimensional (in horizontal plane) saturated incompressible fluid flow in an anisotropic heterogeneous confined aquifer is given by,

$$S \frac{\partial \Phi(x, y, t)}{\partial t} = \frac{\partial}{\partial x} \left(T_{xx}(x, y) \frac{\partial \Phi(x, y, t)}{\partial x} \right) + \frac{\partial}{\partial y} \left(T_{yy}(x, y) \frac{\partial \Phi(x, y, t)}{\partial y} \right) \quad \in \Omega \quad (1)$$

where $T_{xx}(x, y)$ is the transmissivity in x -direction, $T_{yy}(x, y)$ is transmissivity in y -direction, $\Phi(x, y, t)$ is hydraulic or piezometric head, S is the storage coefficient, and Ω is domain of interest.

The transmissivity is related to the hydraulic conductivity by,

$$\begin{aligned} T_{xx}(x, y) &= K_{xx}(x, y)H(x, y) \\ T_{yy}(x, y) &= K_{yy}(x, y)H(x, y) \end{aligned} \quad (2)$$

where $H(x, y)$ is the aquifer thickness at location x and y .

No-flow (Neumann condition) or constant head (Dirichlet condition) are specified on the boundaries of the flow domain, that is,

$$\begin{aligned}
 \frac{\partial}{\partial n} [\Phi(x, y, t)] &= 0 && \text{on } x, y \in \Gamma_1 \\
 \Phi(x, y, t) &= \Phi_0 && \text{on } x, y \in \Gamma_2 \\
 \Phi(x, y, t) &= \Phi(t) && \text{on } x, y \in \Gamma_3
 \end{aligned}
 \tag{3}$$

where $\Gamma_1 + \Gamma_2 + \Gamma_3 = \Gamma =$ boundary of the domain, n is the unit vector normal to the boundary pointing outward, and Φ_0 is the prescribed head.

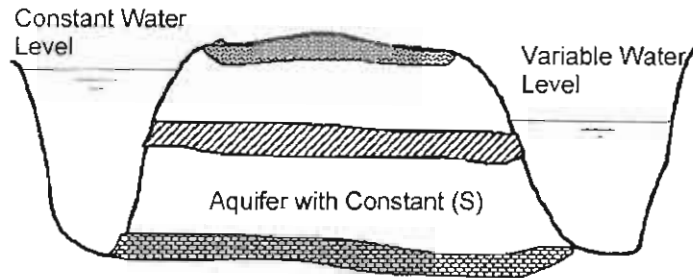


Figure 1. Transient Groundwater Flow in Confined Aquifer.

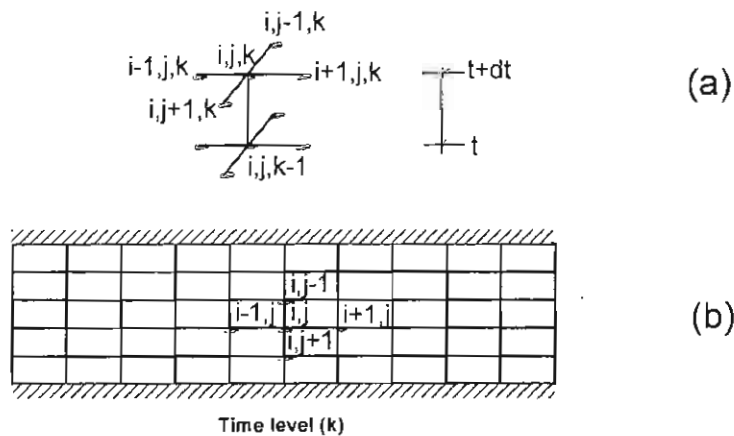


Figure 2. Domain Discretization (b) and The Fully Implicit Numerical Scheme (a).

3. FINITE DIFFERENCE FORMULATION AND SOLUTION BY CONJUGATE GRADIENT METHOD

A finite difference model has been developed for discretization of Eq.(1). A numerical scheme with a five-points operator shown in Figure 2 is used. The finite difference analog for the derivatives are given in the following expressions,

$$T_x(x, y) \left(\frac{\partial \Phi(x, y, t)}{\partial x} \right) \approx T_{x_{i+1/2, j}} \left[\frac{\Phi_{i+1, j}^k - \Phi_{i, j}^k}{\Delta x} \right] \quad (4)$$

where $T_{x_{i+1/2, j}}$ is the interface transmissivity between node $(i+1, j)$ and node (i, j) . This transmissivity could be estimated by the harmonic mean of the surrounding nodes in x -direction,

$$T_{x_{i+1/2, j}} = \frac{2T_{x_{i, j}} T_{x_{i+1, j}}}{T_{x_{i, j}} + T_{x_{i+1, j}}} \quad (5)$$

and $\Phi_{i, j}^k$ is the hydraulic head at node (i, j) at time k .

Similarly,

$$T_y(x, y) \left(\frac{\partial \Phi(x, y, t)}{\partial y} \right) \approx T_{y_{i, j+1/2}} \left[\frac{\Phi_{i, j+1}^k - \Phi_{i, j}^k}{\Delta y} \right] \quad (6)$$

with $T_{y_{i, j+1/2}}$ is given by

$$T_{y_{i, j+1/2}} = \frac{2T_{y_{i, j}} T_{y_{i, j+1}}}{T_{y_{i, j}} + T_{y_{i, j+1}}} \quad (7)$$

Further evaluation leads to

$$\frac{\partial}{\partial x} \left(T_x(x, y) \frac{\partial \Phi(x, y, t)}{\partial x} \right) \approx \frac{T_{x_{i+1/2, j}} \left[\frac{\Phi_{i+1, j}^k - \Phi_{i, j}^k}{\Delta x} \right] - T_{x_{i-1/2, j}} \left[\frac{\Phi_{i, j}^k - \Phi_{i-1, j}^k}{\Delta x} \right]}{\Delta x} \quad (8)$$

$$\frac{\partial}{\partial y} \left(T_y(x, y) \frac{\partial \Phi(x, y, t)}{\partial y} \right) \approx \frac{T_{y_{i, j+1/2}} \left[\frac{\Phi_{i, j+1}^k - \Phi_{i, j}^k}{\Delta y} \right] - T_{y_{i, j-1/2}} \left[\frac{\Phi_{i, j}^k - \Phi_{i, j-1}^k}{\Delta y} \right]}{\Delta y} \quad (9)$$

$$S \frac{\partial \Phi(x, y, t)}{\partial t} \approx S_{i, j} \frac{[\Phi_{i, j}^k - \Phi_{i, j}^{k-1}]}{\Delta t} \quad (10)$$

substitution of Eq.(8), Eq.(9) and Eq.(10) into Eq.(1) leads to the finite difference analog for the partial differential equation as,

$$A_{i,j}\Phi_{i+1,j}^k + B_{i,j}\Phi_{i,j-1}^k + C_{i,j}\Phi_{i-1,j}^k + D_{i,j}\Phi_{i,j+1}^k + E_{i,j}\Phi_{i,j}^{k-1} - F_{i,j}\Phi_{i+1,j}^k = 0$$

where,

$$A_{i,j} = T_{xx,i,j} / \Delta x^2$$

$$B_{i,j} = T_{yy,i,j} / \Delta y^2$$

$$C_{i,j} = T_{yy,i,j} / \Delta x^2$$

$$D_{i,j} = T_{xx,i,j} / \Delta y^2$$

$$E_{i,j} = S_{i,j} / \Delta t$$

$$F_{i,j} = A_{i,j} + B_{i,j} + C_{i,j} + D_{i,j} + E_{i,j}$$

(11)

After the solution of the flow equation, one can calculate the potential head distribution at each time step and consequently the gradient field and the Darcy's velocity field on the grid. This is can be done by differentiation as,

$$q_{x,i,j}^k = -T_{xx}(x,y) \left(\frac{\partial \Phi(x,y,t)}{\partial x} \right) \approx -T_{xx,i,j} \left[\frac{\Phi_{i+1,j}^k - \Phi_{i,j}^k}{\Delta x} \right] \quad (12)$$

$$q_{y,i,j}^k = -T_{yy}(x,y) \left(\frac{\partial \Phi(x,y,t)}{\partial y} \right) \approx -T_{yy,i,j} \left[\frac{\Phi_{i,j+1}^k - \Phi_{i,j}^k}{\Delta y} \right] \quad (13)$$

where $q_{x,i,j}^k$ and $q_{y,i,j}^k$ are the inter nodal Darcy's velocity components between nodes (i,j) and $(i+1,j)$, and between nodes (i,j) and $(i,j+1)$ at time k .

From the Darcy's velocities the pore-velocities are calculated by dividing Eqs.(12) and (13) by the effective porosity of the medium. This is essential to transport models that will be considered in the future.

A large number of solvers are available for systems of linear equations and some of the efficient solvers, in case of heterogeneous systems with large number of nodes, are the iterative ones. All the iterative solvers start with an initial guess of the field variable and in each iteration a new and better approximation is computed. It has been proven that the method of conjugate gradient (CG) is powerful in addressing highly heterogeneous medium. This method is adopted by Elfeki [1996] for steady state flow problems. The CG method is extended in the current study to handle time dependent flow problems. The formulas and the algorithm for implementation in case of transient conditions are presented. The algorithm used here is an extension of the one given by Strikwerda [1989]. Some modifications are adopted to handle the heterogeneity of the medium and transient conditions. A backward difference fully implicit scheme solved by CG is used for the time integration. This technique is fairly simple, completely stable and is free from oscillation problems. The equations to solve are in the form of Eq.(11) which form a linear system $\mathbf{ax} = \mathbf{b}$ where, \mathbf{a} is positive definite matrix and the vector \mathbf{b} contains both zeros and the values of the solution on the boundary. The procedure involves the following steps between two successive time steps k and $k+1$.

First step: an initial iterate $\Phi^{k(0)}$ is given and then the residual $r^{k(0)}$ is computed as,

$$r_{i,j}^{k(0)} = A_{i,j} \Phi_{i+1,j}^{k(0)} + B_{i,j} \Phi_{i,j-1}^{k(0)} + C_{i,j} \Phi_{i-1,j}^{k(0)} + D_{i,j} \Phi_{i,j+1}^{k(0)} + E_{i,j} \Phi_{i,j}^{k-1(0)} - F_{i,j} \Phi_{i,j}^{k(0)} \quad (14)$$

A matrix $P^{k(0)}_{ij}$ is introduced as

$$P_{i,j}^{k(0)} = r_{i,j}^{k(0)} \quad (15)$$

with $|r^{k(0)}|^2$ also being computed by accumulating the products $(r_{i,j}^{k(0)} r_{i,j}^{k(0)})$. In a mathematical form is given by,

$$|r^{k(0)}|^2 = \sum_i \sum_j [r_{i,j}^{k(0)}]^2 \quad (16)$$

Another matrix $Q^{k(0)}_{ij}$ is introduced and computed as

$$Q_{i,j}^{k(0)} = -A_{i,j} r_{i-1,j}^{k(0)} - B_{i,j} r_{i,j-1}^{k(0)} - C_{i,j} r_{i+1,j}^{k(0)} - D_{i,j} r_{i,j+1}^{k(0)} + E_{i,j} r_{i,j}^{k(0)} \quad (17)$$

and the inner product $(P^{k(0)}, Q^{k(0)})$, is computed by accumulating the product $P_{i,j}^{k(0)} Q_{i,j}^{k(0)}$ to evaluate the parameter $\alpha^{k(0)}$ as,

$$\alpha^{k(0)} = \frac{|r^{k(0)}|^2}{(P^{k(0)}, Q^{k(0)})} \quad (18)$$

Note that for Dirichlet boundary condition (prescribed head boundary) $r^{k(m)}$, $P^{k(m)}$, and $Q^{k(m)}$ where m denotes the iteration number, should be zero on the boundary.

Second step: begin the main computation loop. $\Phi_{i,j}^{k(m)}$ and $r_{i,j}^{k(m)}$ are updated by

$$\begin{aligned} \Phi_{i,j}^{k(m+1)} &= \Phi_{i,j}^{k(m)} + \alpha^{k(m)} P_{i,j}^{k(m)} \\ r_{i,j}^{k(m+1)} &= r_{i,j}^{k(m)} - \alpha^{k(m)} Q_{i,j}^{k(m)} \end{aligned} \quad (19)$$

with $|r^{k(m+1)}|^2$ also computed. Another parameter $\beta^{k(m+1)}$ is computed by the formula,

$$\beta^{k(m)} = \frac{|r^{k(m-1)}|^2}{|r^{k(m)}|^2} \quad (20)$$

then P and Q are updated by

$$\begin{aligned} P_{i,j}^{k(m+1)} &= r_{i,j}^{k(m+1)} + \beta_k P_{i,j}^{k(m)} \\ Q_{i,j}^{k(m+1)} &= [E_{i,j} r_{i,j}^{k(m+1)} - A_{i,j} r_{i+1,j}^{k(m+1)} - B_{i,j} r_{i,j-1}^{k(m+1)} - C_{i,j} r_{i-1,j}^{k(m+1)} - D_{i,j} r_{i,j+1}^{k(m+1)}] + \beta_k Q_{i,j}^{k(m)} \end{aligned} \quad (21)$$

and the inner product $(P^{k(m+1)}, Q^{k(m+1)})$ is computed.

Third step: $\alpha^{k(m+1)}$ is computed as the ratio,

$$\alpha^{k(m+1)} = \frac{|r^{k(m+1)}|^2}{(P^{k(m+1)}, Q^{k(m+1)})} \quad (22)$$

and m is incremented.

The conjugate gradient method is terminated when $|r^{k(m)}|$ is sufficiently small. As with the general iterative methods, the method should be continued until the error in the iteration is comparable to the truncation error in the numerical scheme. Table 1 displays the numerical values used to perform the simulations in homogeneous and heterogeneous cases under sudden drop in water level and time series boundary conditions.

Table 1. Simulation Parameters used in Computation

Parameter	Numerical Value
Domain dimensions	200m × 50m
Domain discretization	1.0m × 1.0m
Time step	0.5 day
Upstream Fixed Head Boundary	20 m
Downstream Sudden drop Head Boundary	10 m
Constant Aquifer Thickness	10 m
Homogeneous Hydraulic Conductivity	10 m/day
Heterogeneous Hydraulic Conductivity	1.0, 10., 50., 100. m/day
Accuracy in Computation	0.001
No. of Time Steps	50 Steps (25 days)
Storage Coefficient	0.00001, 0.0001, 0.001, 0.01, 0.1

4. ANALYSIS OF MODEL RESULTS

In the homogeneous aquifer presented in this simulation (Figure 3), it is found that increasing the storage coefficient leads to **smoothing and delaying** in the aquifer response in terms of hydraulic head and Darcy's velocity. The aquifer response in terms of hydraulic head is in phase with the input time series however, the longitudinal Darcy's velocity is out of phase.

In a heterogeneous aquifer, a geological structure with realistic characteristics is generated and displayed in Figure 4. Figure 5 shows the simulation under sudden drop in water level at the right boundary ($S = 0.01$). The Figure shows the propagation of the groundwater head over a record of 9 days when steady state condition is almost achieved. The result in case of time series boundary is displayed in Figure 6. The simulations show different responses according to the value of the storage coefficient. Similar to the homogeneous case, increasing of the storage coefficient leads to smoothing and delaying in the aquifer response in terms of hydraulic head and Darcy's velocity. However, the smoothing effect is more pronounced in the hydraulic head when compared with Darcy's velocity. The aquifer response in terms of hydraulic head field and the lateral Darcy's velocity are in phase with the input time series, however the longitudinal Darcy's velocity is out of phase. Global gradient (regional gradient) magnitude variability coupled with aquifer heterogeneity generates local directional and magnitude gradient variabilities.

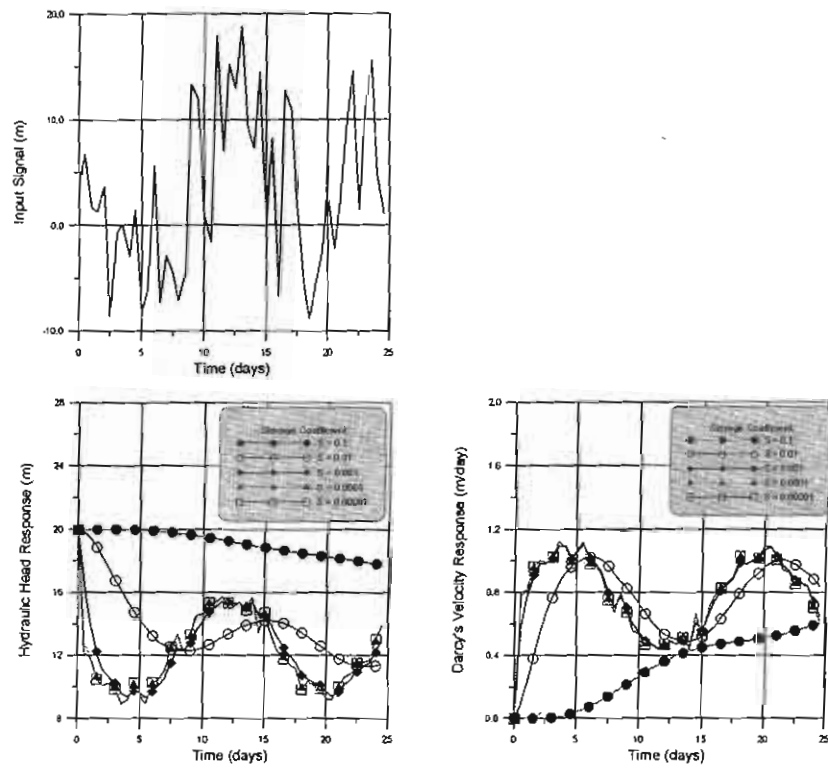


Figure 3. Numerical Simulation in Case of Homogenous Medium under Time Series Boundary at the Water Level (Top left most graph is the input signal, bottom left graph is the aquifer response in terms of hydraulic head at the middle of the aquifer and the bottom right graph is the aquifer response in terms of Darcy's velocity at the same location.

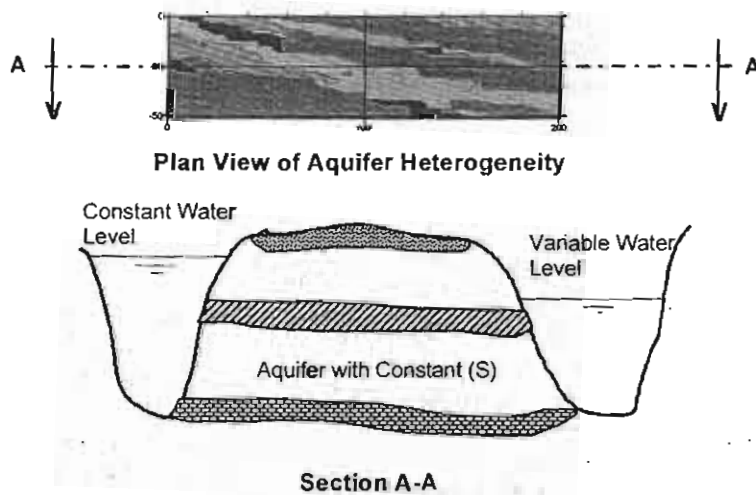


Figure 4. Transient Groundwater Flow in Heterogeneous Confined Aquifer.

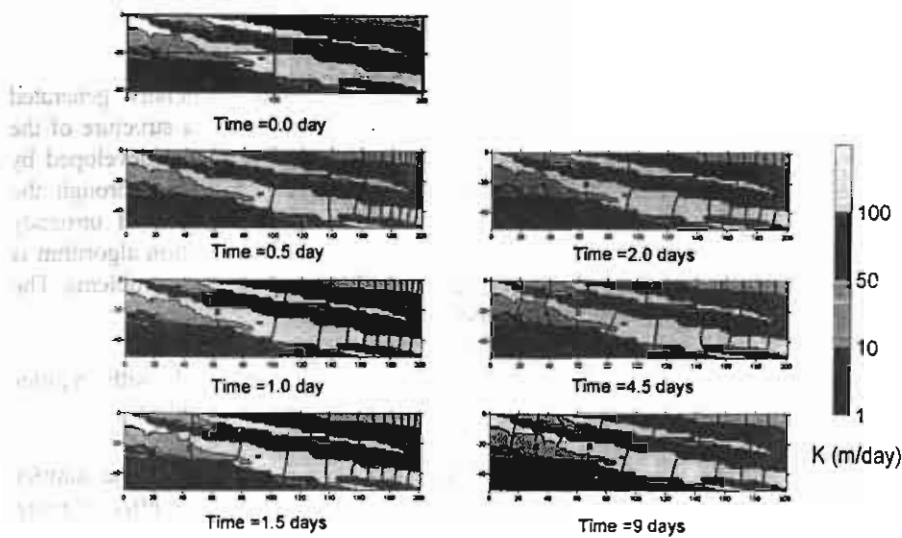


Figure 5. Snapshots of The Hydraulic Head Distribution under Sudden Drop in Water Level in Heterogeneous Confined Aquifer (Steady state condition is almost reached after 9 days).

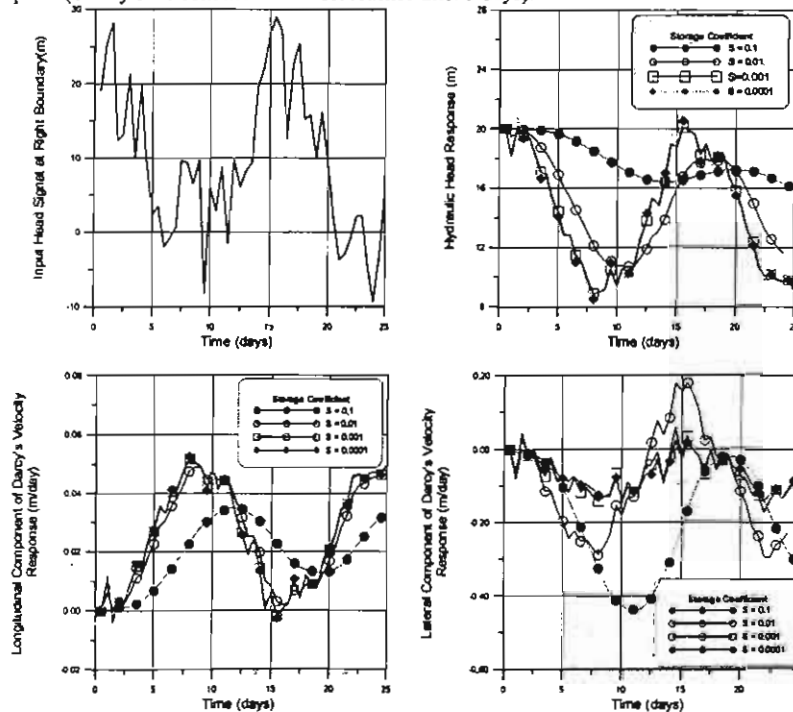


Figure 6. Numerical Simulation Results in Case of Heterogeneous Medium under Time Series Boundary at the Water Level. Top left most graph is the input signal, right most graph is the aquifer response in terms of hydraulic head at the middle of the aquifer, the bottom left graph is the aquifer response in terms of longitudinal Darcy's velocity at the same location and the bottom right graph is the aquifer response in terms of lateral Darcy's velocity at the same location.

5. CONCLUSIONS

Numerical simulations of groundwater flow in homogeneous and artificially generated heterogeneous geological formations have been presented. The heterogeneous structure of the geological patterns has been generated using the coupled chain Markov model developed by Elfeki and Dekking [2001]. Solution of the governing equations is achieved through the application of a finite difference approach to the partial differential equation of unsteady groundwater flow in horizontal plane with heterogeneous properties. The solution algorithm is an extension of the CG method used by Elfeki [1996] to handle time dependent problems. The following conclusions can be drawn from this research:

1. Global gradient (regional gradient) magnitude variability coupled with aquifer heterogeneity generates local directional and magnitude gradient variabilities.
2. Increasing the storage coefficient leads to smoothing and delaying in the aquifer response in terms of hydraulic head and Darcy's velocity. The smoothing effect is more pronounced in the hydraulic heads when compared with Darcy's velocity.
3. In the homogeneous aquifer presented in this simulation, the aquifer response in terms of hydraulic head field is in phase with the input time series however, the longitudinal Darcy's velocity is out of phase.
4. In the heterogeneous aquifer presented in this study, the aquifer response in terms of hydraulic head field and the lateral Darcy's velocity are in phase with the input time series, however the longitudinal Darcy's velocity is out of phase.

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