Date: 18 / 5 / 2014
Time: 3 hours
Full mark: 110 marks

First year
Production Engineering
Second Semester 2013/2014
Mathematics (4) – BAS5121

Mansoura University
Faculty of Engineering
Math. & Eng. Physics Dept.

Exam Guidelines: This Exam contains 4 questions in 2 pages, start every question in a new page.

First part: Complex Analysis

من فضلك إبدأ إجابة هذا الجزء من الجهة اليمنى لورقة الإجابة والجزء الثاني من الجهة اليسرى لورقة الاجابة.

Question (1) [30 pts]

a) [8 pts] Use De Moivre's theorem to prove that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta .$$

b) [8 pts] Solve the equation $z^4 - (1+i) = 0$.

c) [8 pts] Show that $u(x, y) = x\,y^3 - x^3\,y$ is a harmonic function in the whole plane and find its harmonic conjugate $v(x, y)$. Express the function $f(z) = u(x, y) + i\,v(x, y)$ as a function of the complex variable z.

d) [6 pts] Prove that $\lim\limits_{z \to 0} \dfrac{z}{\bar{z}}$ does not exist.

Question (2) [25 pts]

a) [6 pts] Find the region in which the following function is analytic.

$$f(z) = e^x(\cos y + i\sin y).$$

b) [6 pts] Solve the equation $e^z = 2$.

c) [8 pts] Evaluate the complex integral $\oint_C \dfrac{z+1}{z^3(z-1)}\,dz$, where C is the circle C: $|z| = 2$.

d) [5 pts] Evaluate $\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4+2i$ along the curve $C = C_1 \cup C_2$ where C_1 is the vertical straight line from $z = 0$ to $z = 2i$ and C_2 is the horizontal line from $z = 2i$ to $z = 4+2i$.

Question 3 [30 point]

a. **[10 pts]** If solution of $(1-x^2)y''(x) - 2xy'(x) + 20y(x) = 0$ where $y(0) = 3$, $y'(0) = 0$ can be expressed by power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Find

i) The recurrence relation for a_n ii) a_6 , a_5 , a_4 , a_3 , a_2 , a_1 , a_0

b. **[10 pts]** Prove that $\int_0^{\infty} \frac{\cos(x)}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}}$ (Hint: use the relation $\int_0^{\infty} t^{\frac{-1}{2}} e^{-xt} dt = \frac{\Gamma(\frac{1}{2})}{\sqrt{x}}$.

Then multiply both side by $\cos(x)$ and integrate with respect to x from 0 to ∞)

c. **[5 pts]** Find $I = \int_0^{\frac{\pi}{2}} x^2 J_{\frac{1}{2}}^4(x) dx$

d. **[5 pts]** Show that $y(x) = \sqrt{x} J_{\frac{1}{2}}(x)$ is solution of $x^2 y''(x) + (x^2 - 2)y(x) = 0$

Question 4 [25 point]

a. **[10 pts]** Expand the function $f(x) = |x|$ where $-1 \le x \le 1$ in term of Legendre polynomials. Find the first two nonzero terms

b. **[15 pts]** If $f(x) = x - x^2$ where $0 \le x \le 1$. Find

i) Fourier Cosine Series. Then prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

ii) Fourier Sine Series. Then prove that $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$

iii) <u>**Without Integration**</u> if $f(x) = \begin{cases} x - x^2 & 0 \le x \le 1 \\ x^2 - x & -1 \le x \le 0 \end{cases}$. Find

$$I = \int_{-1}^{1} f(x) \sin(5\pi x) \cos(4\pi x) dx$$

<u>**You can use the following relations through the exam:**</u>

$P_0(x) = 1$ $P_1(x) = x$ $P_2(x) = \frac{1}{2}(3x^2 - 1)$ $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} sin(x)$ $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$ $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$ $\Gamma(x)\Gamma(1-x) = \frac{\pi}{sin(\pi x)}$ $0 < x < 1$

$\beta(x,y) = 2\int_0^{\frac{\pi}{2}} sin^{2x-1}(t) cos^{2y-1}(t) dt = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt$ $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

	Mansoura University Faculty of Engineerig First year Production	(اولى إنتاج (تخلفات Time 3 hr. (Date: May, 2014)	Second semester: 2013-2014 Math. 4 - Code: BAS 5121 Final Semester Exam

Answer the following questions [Full Marks 110]

Question 1 [28 Marks]

(a) Prove that: $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$. **[8 marks]**

(b) Evaluate the integrals: (i) $\int_0^1 \sqrt{-\ln x}\, dx$ **(ii)** $\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\, dx$. **[8 marks]**

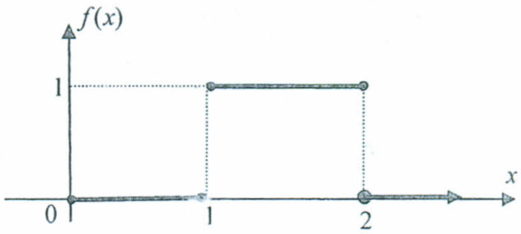
(c) Prove that: $\frac{d}{dx}\left[x^n J_n(x)\right]=x^n J_{n-1}(x)$, **where** $J_n(x)$ **is the Bessel function of order** ***n*****.**

[8 marks]

(d) Write the general solution of the Legendre D. E. $(1-x)^2 y''-2xy'+30y=0$. **[4 marks]**

Question 2 [27 Marks]

(a) Find the Fourier sine series to the function shown in Fig.



[8 marks]

(b) Use the Fourier cosine integral representation to the function $f(x)=e^{-3x}, x>0$, prove that:

$$\int_0^\infty \frac{\omega \sin \omega x}{9+\omega^2}\, d\omega=\frac{\pi}{2}e^{-3x}, x>0.$$ [6 marks]

(c) (i) Use the separation of variables technique; find the solution of the heat equation:

PDE: $U_t(x,t)=kU_{xx}(x,t), \quad 0<x<L, \quad t>0,$

BC: $U(0,t)=U(L,t)=0,$

IC: $U(x,0)=f(x).$

(ii) Find the solution if: $k=1,\ L=2$ and $f(x)$ is the function shown in the Fig. of item (a).

[13 marks]

Good luck (أنظر خلف الورقة) prof. I. El-Kalla

Question3: [28 Points]

a) [6 Points] For any two complex numbers z_1 and z_2, prove that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

b) [6 Points] Describe graphically the region represented by

i. $Re(z) \leq Im(z^2)$, ii. $\left|\frac{2}{z}\right| < 1$.

c) [4 Points] Prove that if $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then each of the functions $u(x, y)$ and $v(x, y)$ are harmonic in D.

d) [6 Points] Find all possible values of $(-1 - i)^{\frac{1}{3}}$ and $(-1)^i$.

e) [6 Points] Solve the equation

$$\cos z = 3.$$

Question 4: [27 Points]

a) [9 Points] Evaluate

$$\oint_{|z|=2} \left(\cosh z + \frac{\sin z}{z} + \frac{ze^z}{z-1} + e^{\frac{\pi}{z}} + \frac{\sinh z}{z(z-i)^2}\right) dz$$

b) [9 Points] Expand $\frac{-1}{(z-1)(z-2)}$ in Laurent series valid for:

i. $1 < |z| < 2$,

ii. $|z| > 2$.

c) [9 Points] Use the residue theorem to evaluate

$$\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{5 + 4\cos\theta}.$$

With my best wishes

Dr. Mohamed Soror