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A NEW TECHNIQUE FOR THE DESIGN OF FIR DIGITAL FILTERS

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ABSTRACT

A method for thynsis of FIR Chebychev digital filters is presented. The approximation in L_∞ norm is obtained making use of lawson's algorithm. This reduces the problem to one of computing certain weighted least square approximations.

I- INTRODUCTION

Several techniques have been developed to design FIR digital filter with prescribed frequency response (1).

By way of example, the techniques required to produce a linear phase digital filter involve; frequency sampling, optimal minimax error techniques and windowing.

Windowing technique has the advantage of simplicity, ease of use and the fact that closed form expressions are often available for the window coefficients (i.e. Fourier series coefficients of its frequency response). The main disadvantage of this technique is that the resulting FIR filters satisfy no known optimality criterion, hence their performance can be considerably improved in most cases (1).

The second design technique of frequency sampling depends on that the desired frequency response should be smooth so that interpolation error between the samples of the frequency response may be small. For band-select filters the frequency response changes sharply across transition bands. The frequency samples occuring in these transition bands are taken as variables and defined by an optimisation algorithm minimising some function of the approximation error of the filter (1).

The third design technique reduces the problem to a Chebychev (minimax) approximation problem. Classicaly, the solution to this problem has implied the selection of the coefficients of a suitable polynomial (or rational function in case of IIR filters) so that it fits the desired frequency response in an optimal equal ripple manner. Remez algorithm method and its generalizations are notable examples of processes for obtaining best approximations using polynomials and rational functions(1)-(4).

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these methods lead to a set of non-linear equations in the filter coefficients. Iterative solution of these equations is much involved and takes along computational time.

In this paper we present a new method for minimax approximation which reduces the computational effort to a great extent. The method is based on the known result that the best minimax approximation may be obtained by computing certain weighted least least squares approximations. Lawson's algorithm consists in generating the required weight functions (5).

II- ALGORITHM FOR MINIMAX APPROXIMATION

We are given a well behaved function $H_d(w)$ describing the desired frequency response of the FIR filter. $H_d(w)$ is specified over the Nyquist frequency interval. The actual frequency response is a linear combination of a sequence of functions $\{g_m\}$

$$H_a(w) = \sum_{m=1}^{M} a_m g_m(w) \qquad \text{.............(1)}$$

The functions $\{g_m\}$ are linearly independent and defined over the Nyquist interval. Their specific forms are given in (1) and depend on the symmetry of the impulse response and the parity of N- the number of samples in the filter impulse response.

It is required to make $H_a(w)$ the best approximation to $H_d(w)$, in the sense that the maximum approximation error over the entire Nyquist interval is minimized. We define an error function e(w) as

$$e(w) = | H_a(w) - H_d(w)| \qquad \text{.............(2)}$$

We also define a norm

$$L_p = \| e \|_p = \left(\int_I |e|^p \, dw \right)^{1/p} \qquad \text{...........(3)}$$

where I is thé set of points at which H_d is defined. A minimization for $\|e\|_p$ defined by Eq. (3) is called a best approximation with respect to L_p. For well behaved functions

$$\max_{i \in I} |e(w_i)| = \lim_{p \to \infty} \left(\int_I |e|^p \, dw \right)^{1/p} \qquad \text{.........(4)}$$

In view of this, minimization in the chebychev (minimax) sense amounts to minimization with respect to L_∞ .

Some techniques have been developed for obtaining sequence of least p^{th} approximations with finite p, which converge in the limit, as p becomes very large, to the minimax solution (2), (6). Recently, however, Lawson has developed another method for minimax approximation, where in the coefficients $\{a_n\}$, which make $H_a(w)$ approximates $H_d(w)$ in the chebychev sense, are obtained through a sequence of iterations (5). At the n^{th} step of iteration, a new set of coefficients a_m^n is calculated such that the function

$$H_a^n (w) = \sum_m a_m^a \quad g_m (w) \qquad \ldots\ldots\ldots\ldots\ldots(5)$$

is the best L_2 (least square)-approximation to the desired response with the error function defined from:

$$\| e \|^2 = \int_I \left| H_a^n (w_i) - H_d(w_i) \right|^2 \quad W_i \, dw \qquad \ldots\ldots\ldots(6)$$

Where W is a weight function. The values W_i of this funotion at the frequencies w_i are calculated from

$$W_i^n = \frac{W_i^{n-1} \left| H_d (w_i) - \sum_{m=1}^{M} a_m^{n-1} \; g_m (w_i) \right|}{\sum_{i \in I} W_i^{n-1} \left| H_d(w_i) - \sum_{m=1}^{M} a_m^{n-1} \, g_m(w_i) \right|} \qquad \ldots\ldots\ldots\ldots(7)$$

We note that

$$\sum_{i \in I} W_i^n = 1 \qquad \ldots\ldots\ldots\ldots(9)$$

The functions $\{ g_m \}$ are not, in general, arthonormal with respect to the weight function W. However, a new set of orthonormal functions $\{ g_m^* \}$ can be obtained from the $\{ g_m \}$ such that

$$g_m^* = \sum_{j=1}^{m} B_{mj} \; g_j \qquad m=1,2,\ldots,M \qquad \ldots\ldots\ldots\ldots(10)$$

and

$$(g_i^*, g_j^*) = \delta_{ij} \qquad \ldots\ldots\ldots\ldots(11)$$

where the inner product of two functions f and g is given by

$$(f, g) = \int_I f(w) \; g(w) . \; W (w) \; dw \qquad \ldots\ldots\ldots\ldots(12)$$

the integration is over the whohe Nyquist interval and δ_{ij} is the Kroncker delta. The B_{mj} are obtained by making use of the Gram-Schmidt procedure (Appendix).

The best approximation to H_d with respect to the norm $\| e \|_2$ defined by Eq.(6) is now given by

$$H_d(w) = \sum_{m=1}^{M} (H_d, g_m^*)\, g_m^* \quad \ldots\ldots\ldots\ldots\ldots(13)$$

Substituting for g_m^* from Eq.(7) and rearranging, an explicit explicit expression for $H_d(w)$ in terms of g_m may now be obtained

$$H_d(w) = \sum_{m}^{M} a_m^n \; g_m \quad \ldots\ldots\ldots\ldots\ldots(14)$$

where

$$a_m^n = \sum_{j=m}^{M} (H_d, g_j^*)\, B_{jk} \quad \ldots\ldots\ldots\ldots\ldots(15)$$

Having determined $\{a_m^n\}$, a new weight function W^{n+1} is calculated using Eq.(7), and a new set of coefficients $\{a_m^{n+1}\}$ are then determined according to the steps indicated above. The iteration process is continued until two successive iterations for the $\{a_m\}$ coefficients do not differ appreciably.

The convergence of the process is proved in (5). However Numerical computations have shown that the convergence is slow without an acceleration scheme. It has been observed that Lawson's algorithm tends to drive the weights to zero everywhere except at the extremal points of the error curve. Hence attempts may be made to speed up the algorithm by making the weights tend to zero as rapidly as possible evergwhere except at the extremal points. The scheme due to Rice and Uson (5) consists in

Step 1: 3 to 4 iterations of Lawson's algorithm are carried out.

Step 2: W_i^n is set to zero, if

$$\left| H_d(w_i) - \sum_m a_m^n \, g_m(w_i) \right| \leq \frac{\| e \|^2}{\max\limits_{i \in I} \left| H_d(w_i) - \sum a_m^n \;\; g_m(w_i) \right|} \quad \ldots(16)$$

Where e is given by Eq.(6)

Step 3: Iteration is started from step 1 again.

III- EXAMPLE

Consider the design of a lowpass linear phase FIR digital filter with cutoff frequency $w_c = \pi/6$. The desired frequency response may be specified by

$$H_d(w_i) = 1 \qquad 0 \leqslant i \leqslant 30$$
$$= 0 \qquad 30 < i \leqslant 180 \qquad \ldots\ldots\ldots\ldots(17)$$

Where

$$i \in I = \{0, 1, 2, \ldots\ldots, 180\}$$

and

$$w_i = \frac{i}{180}\pi \qquad \ldots\ldots\ldots\ldots(18)$$

A valid frequency response of this filter is

$$H_a(w) = \sum_{m=0}^{N-1/2} a_m \text{ Cos } w_m \qquad N\text{-odd} \quad \ldots\ldots\ldots\ldots(19)$$

Where N is the filter length and will be taken equal to 5 in our calculations. The coefficients a_m are related to the filter impulse response h(m) by (1)

$$a(m) = 2h\left(\frac{N-1}{2} - m\right) \quad m = 1, 2, \ldots\ldots, \frac{N-1}{2} \quad \ldots\ldots(20)$$

The functions cos mw are identified with $g_m(w)$ in section II. Accelerated Lawson's algorith is started choosing the initial weight to be equal i.e.,

$$W(w_i) = \frac{1}{181} \qquad i \in I \qquad \ldots\ldots\ldots\ldots(21)$$

The coefficients $\{a_m^1\}$ at the first cycle of iteration are calculated from

$$a_m^1 = 181 \times \frac{2}{\pi} \int_0^{\pi} H_d(w) \text{ Cos}(m\, w)\; W(w)\; dw$$
$$= \frac{2}{\pi} \int_0^{\pi} H_d(w) \text{ Cos }(m\, w)\; dw \qquad \ldots\ldots\ldots\ldots(22)$$

Evidently, there is no need for the Gram. Schmidt procedure at this stage. The iteration cycles are then continued in the manner indicated in section II. The iterations have been found to converge to a third decimal accuracy in 8 iterations. The coefficients $\{a_m\}$ are found to be

$a_o = 1.24 \qquad a_1 = -1.44 \qquad a_2 = 1.24$

The synthesised frequency response is shown in Fig.(1). The corresponding impulse response of the filter is found from

$$h(o) = a_2/2, \quad h(1) = a_1/2, \quad h(2) = a_o$$

Owing to the symmetry of the impulse response,

$$h(3) = h(1), \quad h(4) = h(o).$$

Hence the response of the filter is given by

$$H(z) = 0.62 - 0.72\, z^{-1} + 1.24\, z^{-2} - 0.72\, z^{-3} + 0.62\, z^{-4}$$

Which can be easily realized as in Fig. (2).

IV- CONCLUSION

Use of Lawson's algorithm reduces the problem of minimax approximation to one of weighted least squares approximations. This makes it possible to avoid solving a set of non linear equations in FIR filter coefficients as in other minimax approximation algorithms. Although in the example presented above a filter of odd-length and symmetric impulse response has been considered, the method is readily applicable to filters of even length and symmetric on antisymmetric impulse response.

APPENDIX

The Gram-Schmidt Orthonormolization Procedure

Let g_n designate a set of linearly independent functions of w. It is always possible to construct an orthonormal set of functions g_n as follows.

The first member of the orthonormal set g_1 is

$$g_1^* = \frac{g_1}{(g_1, g_1)^{½}}$$

Where the inner product of two functions f and g is defined by

$$(f, g) = \int f \, g \, W \, dw$$

where W is a weight function and the integration is over the whole interval.

We next construct a function f_2 normal to g_1^* from g_2 and g_1^*. Let

$$f_2 = g_2 - a_1 g_1$$

then require that

$$(g_1^*, f_2) = (g_1^*, g_2) - a_1 (g_1^*, g_1^*) = 0$$

So that

$$a_1 = (g_1^*, g_2)$$

and we find that

$$f_2 = g_2 - (g_1^*, g_2) g_1^*$$

Dividing f_2 by $(f_2, f_2)^{1/2}$ we find the corresponding orthonormal function

$$g_2^* = \frac{f_2}{(f_2, f_2)^{1/2}}$$

The third member of the set is found by similar procedure. We require that $f_3 = g_3 - a_2 g_1^* - a_3 g_2^*$ be simultaneously orthogonal to g_1^* and g_2^* so that

$$(g_1^*, f_3) = (g_1^*, g_3) - a_2 = 0$$

$$(g_2^*, f_3) = (g_2^*, g_3) - a_3 = 0$$

and therefore

$$a_2 = (g_1^*, g_3)$$

$$a_3 = (g_2^*, g_3)$$

and

$$f_3 = g_3 - (g_1^*, g_3) g_1^* - (g_2^*, g_3) g_2^*$$

The orthonormal function is

$$g_3^* = \frac{f_3}{(f_3, f_3)^{1/2}}$$

In general it follows that

$$f_n = g_n - \sum_{k=1}^{n-1} (g_k^*, g_n) g_k^*$$

and

$$g_n = \frac{f_n}{(f_n, f_n)^{1/2}}$$

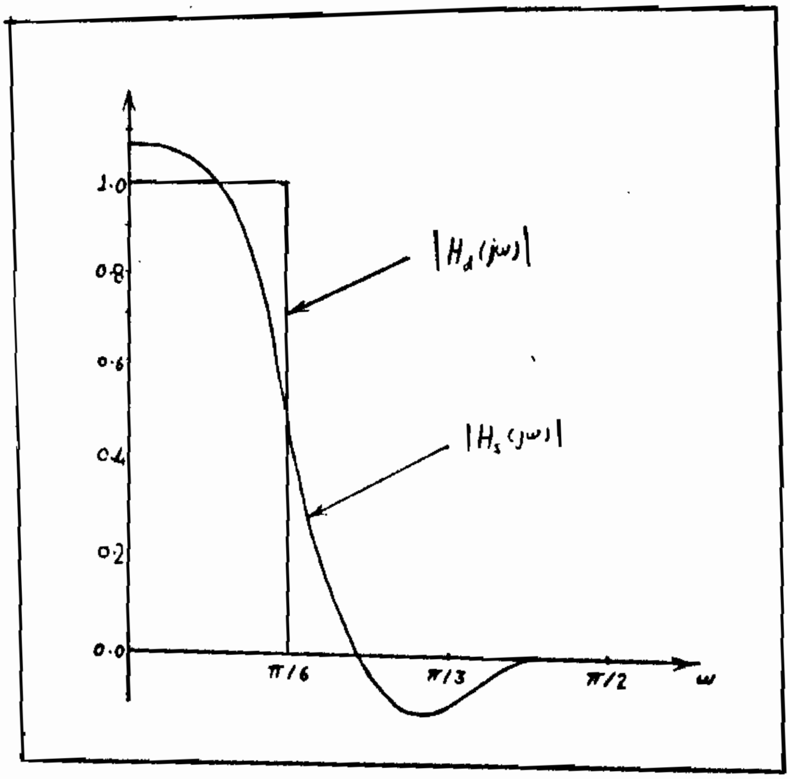

Fig.(1): Magnitude of the desired and synthesised frequency response.

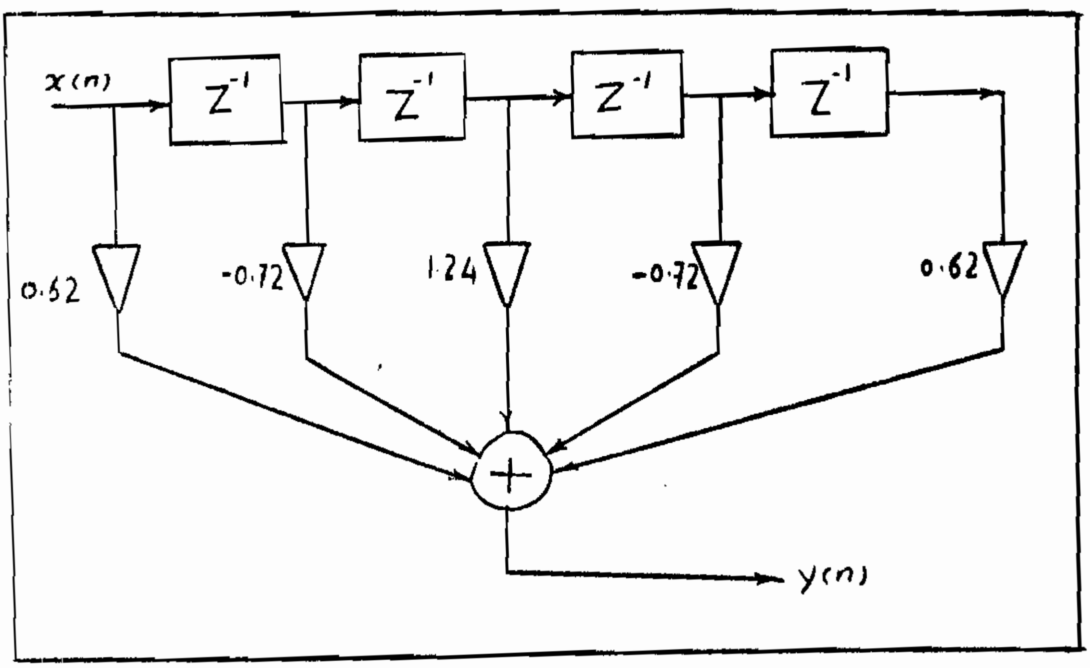

Fig.(2): Realisation of the FIR filter in a direct form.