

A Comparison between Different Topology Optimization Methods

مقارنة بين طرق التصميم التوبولوجي الأمثل المختلفة

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ملخص البحث

يعتبر التصميم التوبولوجي الأمثل من أكثر مجالات التصميم البنائي أهمية. في هذا البحث يتم دراسة التصميم التوبولوجي الأمثل للحصول على أقل مطاوعة للأجزاء الميكانيكية باستخدام طريقة خطوط التقارب المتحركة. هذه الطريقة تعتبر من طرق التصميم الأمثل الشاملة والمرنة، حيث انها تستطيع التعامل مع أى نوع من معادلات الهدف وأى عدد من معادلات القيود. في هذا البحث تم دراسة تأثير تغيير خطوط التقارب الدنيا والقصى على عملية التقارب للتصميم الأمثل بهدف الحصول على التصميم الأمثل بصورة ثابتة وفي أقل وقت ممكن. كما تم تنفيذ التصميم الأمثل لنماذج مختلفة مثل العارضة المثبتة من طرف واحد والعارضة المثبتة من طرفين سواء كانت فى مستوى واحد أو مجسمة ثلاثية الأبعاد. وأيضا تم عمل مقارنة بين طريقة خطوط التقارب المتحركة وطرق مختلفة للتصميم التوبولوجي الأمثل مثل طريقة البرمجة التربيعية المتعاقبة و طريقة المعايير المثلى وطريقة الخلايا ذاتية الحركة الهجينة ، طبقا لقيمة المطاوعة للتصميم والوقت المستهلك لكل طريقة والشكل النهائى.

Abstract

Topology optimization approach is considered among the most interesting fields of structural optimization. In this paper topology optimization for compliance minimization using method of moving asymptotes MMA is presented. This method is considered as a general and flexible optimization method, where it can handle any kind of objective function and any number of constraints. The effect of changing the lower and upper asymptotes on the optimization process convergence is studied for seeking the demanded convergence with more stability and minimum time as possible. Topology optimization of different models such as a cantilever beam and simply supported beam for two and three dimensional structure is accomplished. Also a comparison between Method of Moving Asymptotes (MMA) and different methods such as Sequential Quadratic Programming (SQP), Optimality Criteria (OC), and Hybrid Cellular Automata (HCA) is accomplished according to the compliance value, time consumed and the resulted topological shape.

NOMENCLATURE

x_j : Design variable j

\underline{x}_j : Lower bound of design variable

\bar{x}_j : Upper bound of design variable

P : Penalization factor

L_j : Lower moving asymptote

U_j : Upper moving asymptote

C : Compliance

k_e : Element stiffness matrix

u_e : Element displacement vector

ρ_i : Element density

ρ_0 : Initial density

E_i : Element elasticity

E_0 : The base material elasticity modulus

v_{frac} : Volume fraction

V_0 : Initial volume

n_{elx} : horizontal elements number

n_{ely} : vertical elements number

ν : Poisson's ratio

F: Applied Force on model
U: Global displacement
asy_{int}: Initial asymptotes value
asy_{decr}: Decreased value of asymptotes
asy_{incr}: Increased value of asymptotes
 $\frac{\partial c}{\partial x_i}$: Average compliance sensitivity for element
 and its neighbors

1. INTRODUCTION

Topology optimization approach is considered one of the most interesting fields of structural optimization. It is considered as a promising area that meets a great interest from mechanical designers and manufacturers. It is a relatively new but rapidly expanding research field. It also has important practical applications in automotive and aerospace industries.

Topology optimization strives to achieve the optimal distribution of material within finite volume design domain; which maximizes a certain mechanical performance under specified constraints. Its algorithms selectively remove and relocate the elements to achieve the optimum performance [1]. It can provide a good configuration concept for the structure as a minimum compliance or maximum stiffness design.

The first paper on topology optimization was published over a century ago by the versatile Australian inventor Michell (1904) who derived optimality criteria for the least weight layout of trusses, see [2]. Bendsoe and Kikuchi were presented the landmark paper that had introduced most popular numerical FE-based topology optimization method [3]. Bendsoe had followed that with the method of SIMP which is considered the most popular approach in topology optimization [4].

Ole Sigmund (2001) developed a Matlab code for topology optimization

based on minimizing compliance, mainly using optimality criteria approach that depends on the sensitivity of the objective function [5]. Also there are some methods that can be considered as numerical methods such as, sequential linear programming, sequential quadratic programming, and method of moving asymptotes [6] that can be adopted for topology optimization.

In this paper, the MMA method is applied to different 2D and 3D models (with different number of elements and boundary conditions) as it will be discussed later in section 5&7. A comparison between this method and other methods such as SQP, OC approach [5], and HCA method [1] is presented in section 8.

2. METHOD OF MOVING ASYMPTOTES

The ideal method for structural optimization should be flexible, general, and able to handle not only element size as design variables, but also other variables such as shape and material orientation angles. It should be able to handle all kinds of constraints. MMA method can handle all of these problems in addition to general non-linear programming problems. Moreover it is easy to implement and use. The method of moving asymptotes is a new method for structural optimization that is based on a special type of convex approximation [6].

It is a common approach to mathematical programming method for non-linear optimization problems to formulate a local model at an iteration point. This local model approximates the original one at the given iteration point but is easier to solve. Classical methods like sequential quadratic programming

use such local models. But with respect to the large number of design variables, the use of SQP methods and solving the local models is very costly if not even impossible, due to the fact that gathering second order information for the approximation of the Hessian could be an insuperable task [7].

Consider a structural optimization problem of the following form:

$$P: \min \quad f_0(\mathbf{x}) \quad (\mathbf{x} \in R^n) \quad (1)$$

$$S. t.: f_i(\mathbf{x}) \leq \hat{f}_i, \quad for \ i = 1, \dots, m \quad (2)$$

$$\underline{x}_j \leq x_j \leq \bar{x}_j \quad for \ j = 1, \dots, n \quad (3)$$

Where $\mathbf{x} = (x_1, \dots, x_n)^T$ is the vector of design variables; $f_0(x)$ is the objective function; $f_i(x) \leq \hat{f}_i$ is the behavior constraints; \bar{x}_j and \underline{x}_j are given upper and lower bounds.

The method is interpreted in brief that each $f_i^{(k)}$, is obtained by a linearization of f_i in variables of the type $1/(x_j - L_j)$ or $1/(U_j - x_j)$ dependent on the signs of derivatives of f_i at $\mathbf{x}^{(k)}$, where k is the current iteration. The values of the parameters L_j and U_j are normally changed between the iterations, and we will sometimes refer to L_j and U_j as "Moving Asymptotes". For more details on this method, see [6].

3. TOPOLOGY OPTIMIZATION USING MMA

MMA algorithm that was presented by Krister Svanberg and written with Matlab is used in this section, see [8]. At the beginning, it is important to define the general equations of topology optimization for minimizing the compliance in conjunction with Solid Isotropic Material with Penalization

approach (SIMP) that was presented by Bendsøe [4]. This approach proposed that the material properties are assumed constant within each element in the design domain. Normally, a continuous relative density is used as a design variable. The modulus of elasticity for each element E_i is modeled as a function of the relative density x_i using a power law:

$$\begin{aligned} \rho_i(x_i) &= \rho_0 x_i \\ E_i(x_i) &= E_0 x_i^p, \quad (0 \leq x_i \leq 1) \end{aligned} \quad (4)$$

Where, ρ_i is element density; ρ_0 is the initial density; E_i is element elasticity; E_0 is the elastic modulus of the base material; and p is a penalization power. This power penalizes intermediate densities and drives the design to a black and white structure. To select the proper value of p depend on Poisson's ratio ν , see Bendsøe and Sigmund Material interpolation schemes in topology optimization [9],

$$P \geq \max \left\{ \frac{2}{1-\nu}, \frac{4}{1-\nu} \right\} \quad (\text{In 2D}) \quad (5)$$

$$P \geq \max \left\{ 15 \frac{1-\nu}{7-5\nu}, \frac{3}{2} \frac{1-\nu}{1-2\nu} \right\} \quad (\text{In 3D}) \quad (6)$$

Then the general equations can be written as,

$$\left. \begin{aligned} \min_{\mathbf{x}} : c(\mathbf{x}) &= U^T K U = \sum_{i=1}^n (x_i)^p u_e^T k_e u_e \\ s.t. : \frac{V(\mathbf{x})}{V_0} &= \nu_{frac} \\ : K U &= F \\ : 0 < x_{\min} &\leq x_i \leq 1 \end{aligned} \right\} \quad (7)$$

where, U and F are the global displacement and force vectors, respectively; K is the global stiffness matrix, u_e and k_e are the element displacement vector and stiffness matrix, respectively, \mathbf{x} is the vector of design variables which is relative density of each elements, \mathbf{x}_{\min} is a vector of minimum relative densities (non-zero to avoid singularity) $V(\mathbf{x})$ and V_0 is the

material volume and the initial volume respectively; where v_{frac} is the prescribed volume fraction.

If it is desired to optimize a 2D model, then the model will be discretized to horizontal elements number n_{elx} , and vertical elements number n_{ely} as in the initial design in Fig. 1. Then the number of design variables that will be used, n equals to $n_{elx} * n_{ely}$, also the number of constraints equations, m equals to **2** as in equation 7.

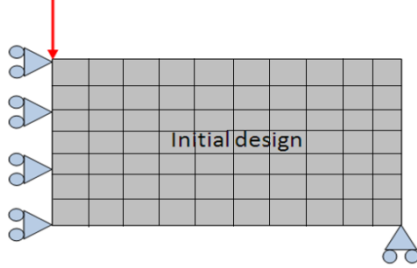


Fig. 1. Initial design domain of half MBB beam

Using a Matlab 2D finite element analysis with the properties of model as shown in Table.1, the optimal topology design will be obtained as shown in Fig. 2.

Table. 1. Design parameters for topology optimization of half MBB beam	
Properties	Values
Young's modulus (E)	1 N/mm ²
Poisson's ratio (ν)	0.3
Force (F)	1 N
SIMP factor (P)	3
Volume fraction (v_{frac})	0.5
No. of elements ($n_{elx} * n_{ely}$)	30*20
Initial design variables \mathbf{x}_0	0.5

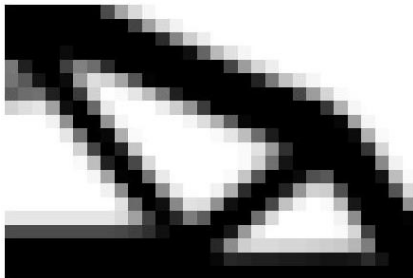


Fig. 2. The resulted topology optimization of half MBB 30x20 element using MMA method

It is observed that with increasing the number of elements, the method will consume more time and it will be impractical. Consequently changing the range of lower and upper moving asymptotes (L_j, U_j) closer or far away from the design variables x_j , the convergence process can be achieved. More details on how to choose the moving asymptotes and how to generate strictly conservative or more linearly approximations can be found in [6]. This problem will be treated in next section.

4. EFFECT OF LOWER AND UPPER MOVING ASYMPTOTES ON OPTIMIZATION CONVERGENCE

Since the method of moving asymptotes is a general method, so the asymptotes can be adopted to be suitable for seeking the demanded convergence of specific problems. A general (although heuristic) rule for how to change the values of $L_j^{(k)}$ and $U_j^{(k)}$ is the following:

- a) If the process tends to oscillate, then it needs to be stabilized. This stabilization may be accomplished by moving the asymptotes closer to the current iteration point.
- b) If, instead, the process is monotonic and slow, it needs to be relaxed. This may be accomplished by moving the asymptotes away from the current iteration point, See [6].

The default rules for updating the lower asymptotes $L_j^{(k)}$ and the upper asymptotes $U_j^{(k)}$ will be now explained. The first two iterations, when $k=1$ and $k=2$; will be:

$$\begin{aligned} L_j^{(k)} &= x_j^{(k)} - asy_{int} (x_j^{max} - x_j^{min}) \\ U_j^{(k)} &= x_j^{(k)} + asy_{int} (x_j^{max} - x_j^{min}) \end{aligned} \quad (8)$$

In later iterations, when $k \geq 3$

$$\begin{aligned} L_j^{(k)} &= x_j^{(k)} - \gamma_j^{(k)} (x_j^{(k-1)} - L_j^{(k-1)}) \\ U_j^{(k)} &= x_j^{(k)} + \gamma_j^{(k)} (U_j^{(k-1)} - x_j^{(k-1)}) \end{aligned} \quad (9)$$

Where,

$$\gamma_j^{(k)} = \begin{cases} asy_{decr}, & \text{if } (x_j^{(k)} - x_j^{(k-1)})(x_j^{(k-1)} - x_j^{(k-2)}) < 0 \\ asy_{incr}, & \text{if } (x_j^{(k)} - x_j^{(k-1)})(x_j^{(k-1)} - x_j^{(k-2)}) > 0 \\ 1, & \text{if } (x_j^{(k)} - x_j^{(k-1)})(x_j^{(k-1)} - x_j^{(k-2)}) = 0 \end{cases} \quad (10)$$

Where the default value of asy_{int} equals 0.5, asy_{decr} equals 0.7 and asy_{incr} equals 1.2, see [11]. It is also found that there are some rules that can be used in the sub-problem file in the MMA code, see [8]. That can be added to the previous asymptotes rules. These rules are:

$$\begin{aligned} L_{j\ min}^{(k)} &= x_j^{(k)} - S_{max} (x_j^{max} - x_j^{min}) \\ L_{j\ max}^{(k)} &= x_j^{(k)} - S_{min} (x_j^{max} - x_j^{min}) \\ U_{j\ min}^{(k)} &= x_j^{(k)} + S_{min} (x_j^{max} - x_j^{min}) \\ U_{j\ max}^{(k)} &= x_j^{(k)} + S_{max} (x_j^{max} - x_j^{min}) \\ L_j^{(k)} &= \max(L_{j\ min}^{(k)}, L_{j\ max}^{(k)}) \\ L_j^{(k)} &= \min(L_{j\ min}^{(k)}, L_{j\ max}^{(k)}) \\ U_j^{(k)} &= \min(U_{j\ min}^{(k)}, U_{j\ max}^{(k)}) \\ U_j^{(k)} &= \max(U_{j\ min}^{(k)}, U_{j\ max}^{(k)}) \end{aligned} \quad (11)$$

Where the default values of S_{max} and S_{min} is 10 and 0.01. These values can be changed to suit any optimization problem.

To illustrate the difference between topology optimization using different ranges of lower and upper asymptotes. An example of half MBB beam with 20x10 elements is implemented with different values of S_{max} and S_{min} in equation (11) such as (100, 0.01) and (600, 0.06) for example. Fig. 3 shows the convergence of topology optimization process at first case which shows that the convergence is steady

after 440 iteration number, with compliance $C = 96.8030$ N.mm.

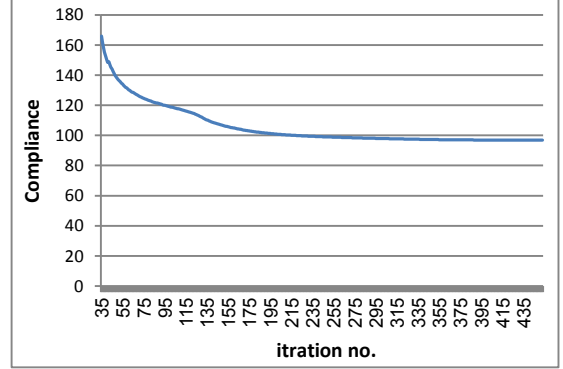


Fig. 3. Convergence of 20x10 elements half MBB beam with S_{max} and S_{min} of 100 and 0.01.

Fig. 4 shows the optimization process at the second case and it indicates that there is no convergence (it finally oscillates between two values of compliance $C= 99$, $C= 100$) and does not introduce the optimum solution.

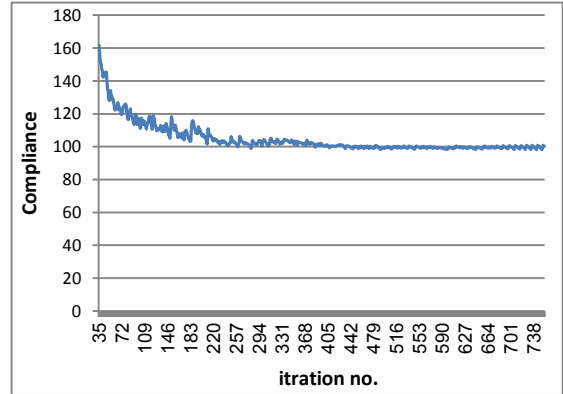


Fig. 4 Divergence of 20x10 elements half MBB beam with S_{max} and S_{min} of 600 and 0.06.

5. TOPOLOGY OPTIMIZATION OF TWO DIMENSIONAL (2D) MODELS

5.1. Cantilever Beam

The initial values that are introduced in Table.1 with a model 30x20 mm and the boundary conditions as shown in

Fig. 5 are used. The final topological optimum design is obtained as shown in Fig. 6 with a minimum compliance of 36.76 N.mm while the compliance at the beginning equals to 154.87 N.mm and the volume fraction v_{frac} was 0.5 from the total volume.

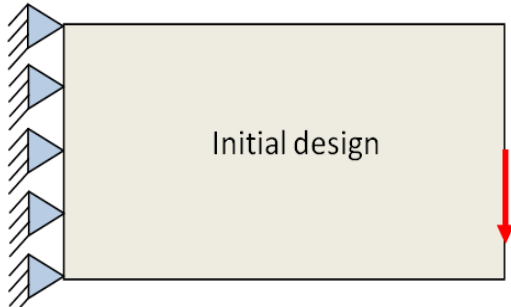


Fig. 5. Initial design of cantilever beam 30x20 elements

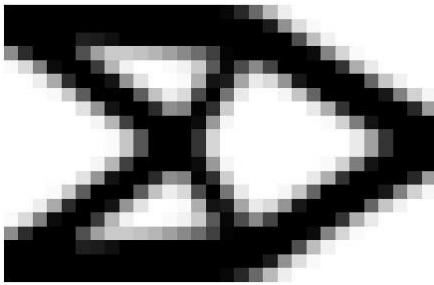


Fig. 6. Topology design of cantilever beam of 30x20 elements

Topology optimization of a cantilever beam with different mesh size is summarized in Fig. 7 which shows that the finer mesh leads to a topological optimum design with less compliance and finer shape than coarser mesh. Although it takes longer time, it has a finer shape with higher resolution and easy to determine the void and material areas.

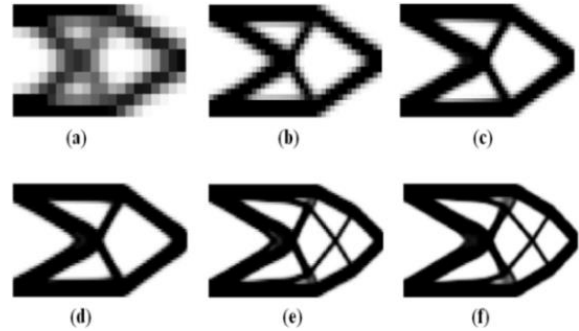


Fig. 7. Topology optimization for different mesh size for cantilever (a) 20x10 elements with compliance $C=88.8544$ (b) 40x20 elements $C=69.0953$ (c) 60x30 elements $C=66.6591$. (d) 80x40 elements $C=65.0711$ (e) 100x50 elements $C=65.1185$ (f) 120x60 elements $C=64.9388$

5.2. Half MBB Beam

A topology optimization of MBB beam with different mesh size with the same initial design as shown in Fig. 1 and initial design parameters as in Table.1 is illustrated in Fig. 8. Same conclusions are reached as in section 5.1.

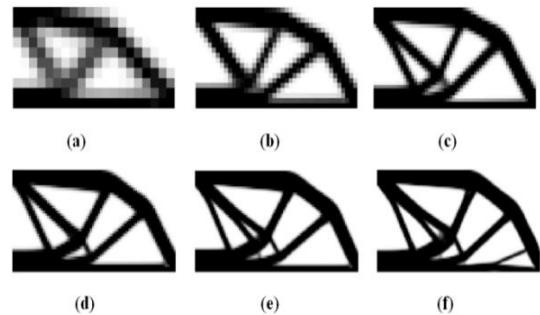


Fig. 8. Topology optimization for different mesh size of half MBB (a) 20x10 elements with compliance $C=96.9919$ (b) 40x20 elements $C=84.1450$ (c) 60x30 elements $C=81.3742$ (d) 80x40 elements $C=80.0763$ (e) 100x50 elements $C=79.6268$ (f) 120x60 elements $C=79.8129$.

6. IMPROVEMENT OF NUMERICAL INSTABILITIES

Using the mesh independency filtering developed by (Peterson and Sigmund 1998) [12], the filter modifies the design sensitivity of a specific element based on a weighted average of

the element sensitivities in a fixed neighborhood.

Where the sensitivity of the objective function is found as:

$$\frac{\partial c}{\partial x_i} = -p(x_i)^{p-1} u_i^T k_i u_i \quad (12)$$

Modifying the element sensitivities using the mesh-independency filter to involve the effect of sensitivity of neighbors [5] is found as:

$$\frac{\partial \hat{c}}{\partial x_i} = \frac{1}{x_i \sum_{a=1}^N \hat{H}_a} \sum_{a=1}^N \hat{H}_a x_a \frac{\partial c}{\partial x_a} \quad (13)$$

$$\hat{H}_a = r_{min} - dist(i, a)$$

Where \hat{H}_a is the convolution operator (weight factor); and the operator $dist(i, a)$ is defined as the distance between center of element i and center of element a . The convolution operator \hat{H}_a is zero outside the filter area. Effect of using mesh independency filter is shown in Fig. 9 which is a cantilever with mesh of 60x30 elements.



Fig. 9. Difference between topological design (a) without mesh filtering and $C= 64.6405$. (b) with mesh filter and $C= 66.6591$.

7. TOPOLOGY OPTIMIZATION OF THREE DIMENSIONAL (3D) MODELS

7. 1. 3D Cantilever Beam

the method of moving asymptotes (MMA) is applied on three dimensional models, and using ANSYS finite element analysis with "Solid45" linear and isotropic solid element type. An

interface between the ANSYS program and the MMA Matlab code is achieved.

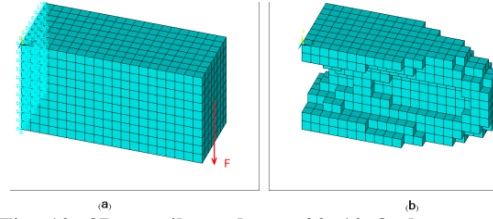


Fig. 10. 3D cantilever beam 20x10x8 elements (a) initial design (b) final topology optimization design

Using a different mesh size (i.e. solid model of length*width*thickness, 20x10x8) as shown in Fig. (10-a) where the initial design and boundary conditions (loads and DOF) is illustrated, and the final topological optimization design is shown in Fig. (10-b). Also Fig. 11 shows topological optimum design for the same initial design but with different mesh size of 30x20x10.

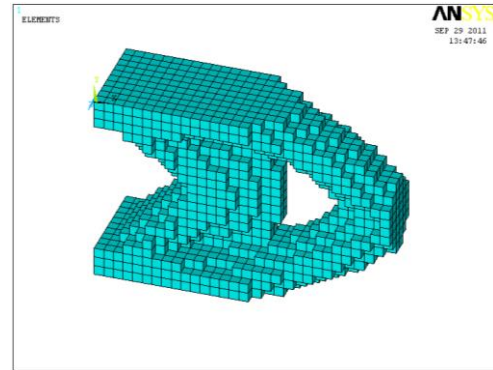


Fig. 11. Final topology optimization of 3D cantilever beam 30x20x10 elements.

7. 2. 3D Half MBB Beam

The method of moving asymptotes is also applied on a three dimensional half MBB beam. Applying MMA algorithm on this model using a Matlab code written by Krister Svanberg and the finite element analysis using ANSYS program. With elements number 20x10x8 as shown in Fig (12-a), as the initial design and boundary conditions (loads and DOF) is illustrated. The final topological optimization design is illustrated in Fig. (12-b), where the

compliance at first iteration is $C= 63.09$ N.mm and at final optimum iteration became $C= 11.57$ N.mm. Also Fig. 13 shows the topology optimization of half MBB beam but with number of elements of $30 \times 20 \times 10$, the compliance of this example at the first iteration was $C= 42.91$ N.mm and at final iteration becomes $C= 6.81$ at the same volume fraction $v_{frac} = 0.5$.

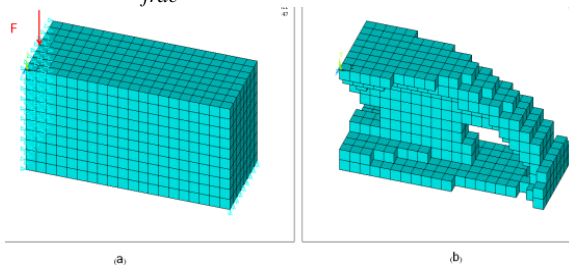


Fig. 12. 3D MBB beam 20x10x8 elements (a) initial design (b) final topology optimization design

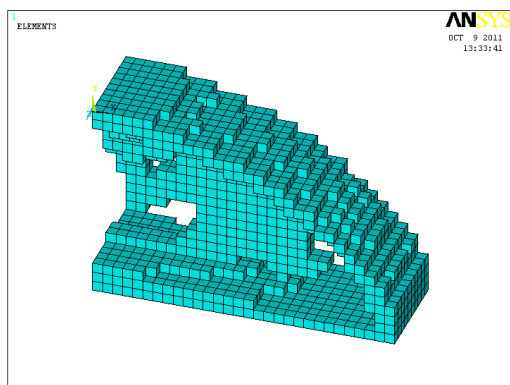


Fig. 13 Final topology optimization of 3D half MBB beam 30x20x10 elements

8. COMPARISON BETWEEN MMA, SQP, OC, AND HCA METHODS

In this section, a comparison between MMA method and different methods such as SQP, OC used in [5] and HCA with proportional integral derivative (PID) control rule [1] presented by Tovar et al. is accomplished. This comparison is achieved according to the compliance values, the time consumed



































and the resulted topological shape in each method.

All topology optimization methods use the same initial design and the same conditions in order to achieve a fair comparison (i. e. the same 2D finite element analysis using Matlab). It is important to define the type of PC that is used and its CPU and memory specifications to make time readings more realistic. HP computer with INTEL Core 2 Duo CPU 2.01GHZ and 2 GB of RAM is used here. Table.2 shows the difference between each of these methods in compliance, time and resulted shape.

Fig. 14 shows that the compliance values in each method almost near and equal except the HCA method which shows that with increasing the number of elements, the compliance increase than other methods. It indicates that this method will not be a practical method. Fig. 15 shows a large difference in the time consumed from one method to another. At the beginning with a small number of elements, it is observed that the time consumed in all methods is almost equal. But with increasing the number of elements, the method of SQP results in a large increase in the consumed time which makes this method totally impractical.

MMA method comes the second method in consumed time after SQP method, as shown in Fig. 15. The time that MMA method consumes with 30×30 elements is 20 minutes, while SQP method consumes 199 minutes. While at 50×50 elements MMA method consumes 80 minutes, despite SQP method that consumes more than 25 hours and does not give the final optimum solution because it becomes out of memory of CPU.

Table.2 Difference in time consumed, compliance, and resulted shape between MMA, SQP, OC, and HCA methods

No. of elements	comparison	MMA method	SQP method	OC method	HCA method
10 x 10	Compliance	24.6004	23.9268	24.575	20.4345
	Time:	1 min. 36 sec.	21 sec.	2 sec.	2 sec.
	Figure:				
20 x 10	Compliance	96.863	77.87	97.32	67.4349
	Time:	1 min. 50 sec.	10 min. 46 sec.	7 sec.	14 sec.
	Figure:				
20 x 20	Compliance	25.094	25.844	24.816	24.9139
	Time:	1 min. 10 sec.	9 min. 14 sec.	5 sec.	3 sec.
	Figure:				
30 x 10	Compliance	224.909	852.991	225.64	187.8579
	Time:	6 min. 21 sec.	5 hrs. 30 min.	7 sec.	8 sec.
	Figure:				
30 x 20	Compliance	47.854	46.2178	47.088	42.6086
	Time:	10 min. 29 sec.	3 hrs. 30 min.	19 sec.	4 sec.
	Figure:				
30 x 30	Compliance	25.6280	25.3993	25.6661	29.001
	Time:	20 min. 50 sec.	3 hrs. 19 min.	47 sec.	7 sec.
	Figure:				
40 x 30	Compliance	38.5279	43.7584	38.3422	39.5394
	Time:	15min. 30 sec.	20 hrs. 2 min.	40 sec.	11 sec.
	Figure:				
40x40	compliance	26.1935	26.8296 not final itr.	26.2098	30.9159
	Time:	40 min. 23 sec.	Over 25 hrs.	3 min. 33 sec.	9 sec.
	Figure:		Not found		
50x50	compliance	26.7830	Out of memory of CPU	26.6650	31.5705
	Time:	1 hrs. 20 min.		1 min. 48 sec.	1 min. 33 sec.
	Figure:				

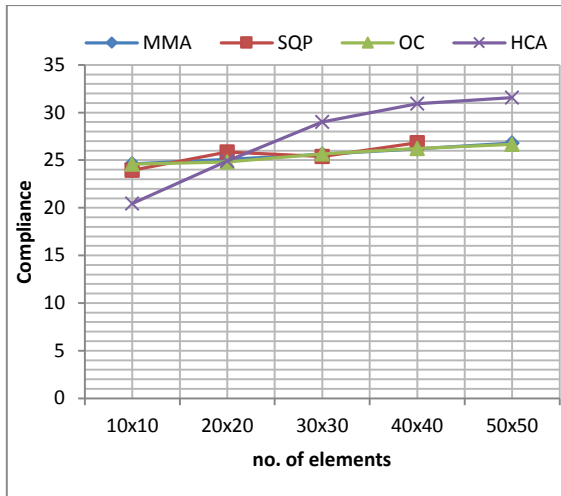


Fig. 14. Difference in compliance between MMA, SQP, OC, and HCA

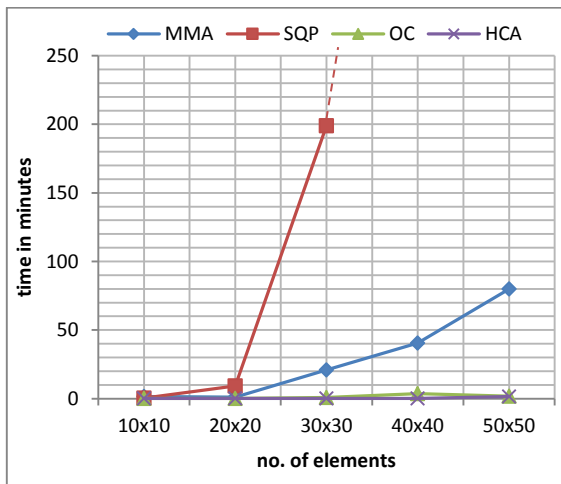


Fig. 15 Difference in time consuming between MMA, SQP, OC, and HCA

Also, if we wish to compare the HCA with OC method, it is observed that HCA is faster than the last one in convergence and reaching the optimum solution and gives minimum compliance values for the small number of elements but with increasing this number we can notice from Fig. 14 that HCA gives high compliance values than other methods.

The HCA and OC methods considered as non general optimization methods and cannot be

used for any optimization problem, while the SQP and MMA methods are considered as general optimization methods and handle any objective function and any number of constraints.

If we wish to change the objective function from the compliance minimization to any other objective function in this case, MMA method will be the most suitable and general optimization method.

9. CONCLUSIONS

- This paper has investigated the topology optimization using the method of moving asymptotes and other methods and shows the difference between them.
- The method of moving asymptotes is considered as a general and flexible method for structural topology optimization problems, it can handle any type of optimization problems. This paper shows that the MMA is the most convenient optimization method for any type of objective function and any number of constraints equations and also reaches the optimum solution with a minimum time.
- Using the method of moving asymptotes let the one control the convergence, stability and speed of the optimization process.
- With increasing the number of elements, the range of asymptotes should be increased to save much time. The recommended range of asymptotes that can be make the convergence stable as indicated in eqn. 11 is from 100 to 1200.

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