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on the locality and hausdorff stability of the compensated convex transformations for functions in the hölder space

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Abstract: The subject of the present work deal with the issue of filling the large gap between bounded Hölder function and its envelope. For this purpose, the compensated convex transformation is used to obtain the radius of the locality property for Hölder functions. The Hausdorff stability for n-compensated convex transformations for Hölder functions is confirmed.

keywords: Bilal distribution; censored data; parameters estimation; lifetime performance index.

1.Introduction

The general (lower, upper and mixed) compensated convex transformations (For short, CCT) are a tight approximation method of functions. Zhang [8] presented the concept of the general CCT and their properties such as the locality property. The locality property of the CCT for bounded functions was presented by Zhang et al. [15]. Also, the locality property of the CCT for Lipschitz-continuous functions was presented by Zhang et al. [11]. The importance of this transformations appears in many fields such as singularity extraction, image processing and the interpolation, see [5, 13]. It is notice from the work of Zhang [8] that the primary tool for studying the general CCT of a given function is the convex envelope of the function. Since the convex envelope plays an important role, hence, Carathéodory's theorem has the same importance where it gives an analytical definition of convex envelope, see [15]. The Hausdorff stability results for quadratic CCT for bounded uniformly continuous positive function with respect to closed sample sets has been presented in [12], and also Hausdorff stability was established when the functions is bounded with respect to the Hausdorff distance between graphs of the sampled functions [1]. Other studies on this topic may be found in References, see [9, 10, 14]. From the above mentioned references, it is noted that, there is a few published studies on this subject, particularly, for the case of Hölder functions. In the present work, the Hausdorff stability for n-upper compensated convex transformations is studied to fill the large gap between bounded Hölder functions and its envelopes. Moreover, the radius of locality of the quadratic compensated convex transformation is obtained for Hölder function.

2. Preliminaries

We will introduce some basic definitions, notations and well-known results that we are going to use during this paper and for further details, see [4, 6, 7, 8, 15]. Through this paper

η denotes the convex function parameter

x denotes the vector$(x_1, x_2, \dots\dots, x_2) \in \mathbb{R}^n$.

Definition 2.1. Consider the function

$f : \mathbb{R}^n \to \mathbb{R}$ satisfies for positive constants m, d and η may be called a convex function parameter.

$f(\mathbf{x}) \geq -m|\mathbf{x}|^q - d$, for all $\mathbf{x} \in \mathbb{R}^n$, (2.1)

then the q-lower CCT for $q > 1$ is defined as. (2.2)

Also, we can rewrite condition (1.1) in another form such as $f(\mathbf{x})$ at $\mathbf{x}$ is concave down. Similarly, if the function f satisfies

$f(\mathbf{x}) \leq m|\mathbf{x}|^q + d$, for all $\mathbf{x} \in \mathbb{R}^n$, 2.3)

then the q- upper CCT for $q > 1$ is defined as.

$C^u_{q,\eta}(f)(\mathbf{x}) = \eta|\mathbf{x}|^q - \mathbf{CO}[\eta|\cdot|^q - f](\mathbf{x}), \eta > m$.(2.4)

For the mixed CCT, we can deduce from Definition 1.1 that

$$C^u_{q,\eta}(f)(\mathbf{x}) = -C^l_{q,\eta}(-f)(\mathbf{x}),$$

when f satisfies $|f(x)| \leq m|\mathbf{x}|^q + d$, for all $\mathbf{x} \in \mathbb{R}^n$ and the q-mixed CCT is given by

$$C^{u,l}_{q,\eta,\xi}(f(\mathbf{x})) = C^u_{q,\eta}\left(C^l_{q,\xi}(f)(\mathbf{x})\right), \qquad (2.5)$$
$$C^{l,u}_{q,\eta,\xi}(f(\mathbf{x})) = C^l_{q,\eta}\left(C^u_{q,\xi}(f)(\mathbf{x})\right),$$

where $\eta > m$.

As a particular case, when $q = 2$ in Definition 1.1, we can define the so-called quadratic CCT and we write C^u_η, C^l_η for upper and lower CCT, respectively.

Algorithm 2.1. The technique of compute the q-lower CCT of $f(\mathbf{x})$ satisfies condition (2.1) can be written as in the following algorithm. (A similar technique for the q-upper CCT can be presented).

Step1. Add to $f(\mathbf{x})$ a weight function $\eta|\mathbf{x}|^q$and let it$g(\mathbf{x})$.

Step2. Find the first derivative of$g(\mathbf{x})$.

Step3. Solve the equation $g'(\mathbf{x}) = 0$ to find the critical points.

Step4. Find from the previous step, the global minimum of $g(\mathbf{x})$ let's say $g(\mathbf{x}_0)$.

Step5.

$$C^l_{q,\eta}(f)(\mathbf{x}) = \begin{cases} g(\mathbf{x}_0) - \eta|\mathbf{x}|^q & \text{if } |\mathbf{x}| \leq \mathbf{x}_0, \\ f(\mathbf{x}) & \text{if } |\mathbf{x}| \geq \mathbf{x}_0. \end{cases}$$

As a particular case:

If $f(\mathbf{x})$ is lower semi continuous function

1. Find the equation of tangent line of right side.

2. Find the intersection point of the equation of the curve in right side and the equation of the tangent line.

3. Find the equation of straight line between intersection point in right side that we get and intersection point of left side with coordinates.

Definition 2.2. Affine function is a single valued mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\eta\mathbf{x} + (1-\eta)\mathbf{y}) = \eta T(\mathbf{x}) + (1-\eta)T(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$and $\eta \in \mathbb{R}$.

Definition 2.3. (Affine Hull) of the set $\mathbf{S} \subseteq \mathbb{R}^n$ is the smallest affine set containing $\mathbf{S}$ and denoted by $\mathrm{Aff}(\mathbf{S})$.

Definition 2.4. Consider the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. The convex envelope of f at $\mathbf{x_0} \in \mathbb{R}^n$ is given by

$$\mathbf{CO}[f](\mathbf{x}_0) = \sup_{\mathbf{y}\in\mathbb{R}^n}\{\ell(\mathbf{x}_0) : \ell \in \mathrm{Aff}(\mathbb{R}^n) : \ell(\mathbf{y}) \leq \mathrm{f}(\mathbf{y})\} \text{ for all } \mathbf{y} \in \mathbb{R}^n. \quad (2.6)$$

Following Caratheodory's theorem, we deduce the local convex envelope of the function f at $\mathbf{x_0}$ for $\eta_i, i = 1,2,\dots,n$ can also given by

$$\mathbf{CO}[f](\mathbf{x}_0) = \inf_{1\leq i\leq n+1}\{\textstyle\sum_{i=1}^{n+1}\eta_i f(\mathbf{x}_i) : \sum_{i=1}^{n+1}\eta_i = 1, \sum_{i=1}^{n+1}\eta_i\mathbf{x}_i = \mathbf{x}_0, \mathbf{x}_i \in \mathbb{R}^n\}. \quad (2.7)$$

In fact, the weight function that we use be a coercive function.

Definition 2.5. The function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be coercive or supper linear growth function if $\frac{f(\mathbf{x})}{|\mathbf{x}|} \to \infty$ as $|\mathbf{x}| \to \infty$.

The CCT satisfies many properties such as translation invariant, Rotation, continuity, monotonicity preserving and locality, see [8,15]. Let's begin with the (**Translation Invariant Property**).

Proposition 2.1. Consider the function

$f : \mathbb{R}^n \to \mathbb{R}$ satisfies (2.1), then the q-lower CCT for $q > 1$ is translation-invariant against the weight function, that it means

$$C^l_{q,\eta}(f)(\mathbf{x}) = \mathbf{CO}[\eta|(\cdot) - \mathbf{x}_0|^q + f](\mathbf{x}) - \eta|\mathbf{x} - \mathbf{x}_0|^q.$$

Similarly, for the q-upper CCT, if f satisfies (2.3) then we have

$$C^u_{q,\eta}(f)(\mathbf{x}) = \eta|\mathbf{x} - \mathbf{x}_0|^q - \mathbf{CO}[\eta|(\cdot) - \mathbf{x}_0|^q - f](\mathbf{x}),$$

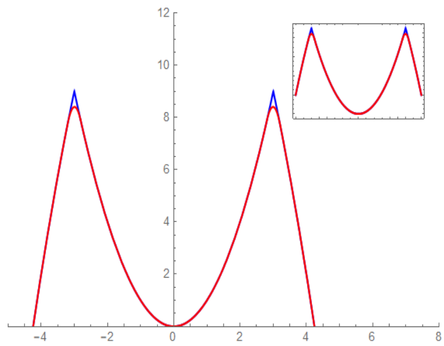
for all $\mathbf{x} \in \mathbb{R}^n$ and for every fixed $\mathbf{x_0}$.

In the following we give an example of function of one variable and we compute the lower CCT.

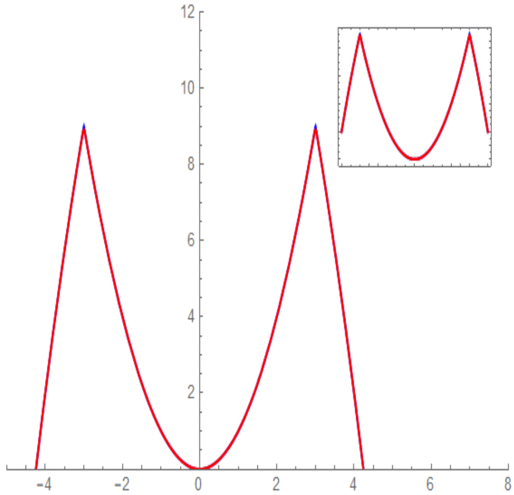
Example 2.1. Consider $f(x) = -|x^2 - 9| + 9, x \in \mathbb{R}$. For $\eta > 0$, we have $f(x)$ has two singular point at $x = \pm 3$. Since $f(\mathrm{x})$ satisfies the condition (2.1), then by using Algorithm 1.1 the lower CCT is given as follows

$$C^l_\eta(f)(x) = \begin{cases} 9 + \dfrac{3}{\eta}x - \eta(x+3)^2 & \text{if } \dfrac{-3\eta}{\eta-1} \leq x \leq \dfrac{-3\eta}{\eta+1}, \\ 9 - \dfrac{3}{\eta}x - \eta(x-3)^2 & \text{if } \dfrac{3\eta}{\eta+1} \leq x \leq \dfrac{3\eta}{\eta-1}, \\ 18 - x^2 & \text{if } |x| \geq \dfrac{3\eta}{\eta-1}, \\ x^2 & \text{if } |x| \leq \dfrac{3\eta}{\eta+1}. \end{cases}$$

Fig 1, displays the graph of the lower CCT.



(a)



(b)

Fig 1: The lower CCT for Example 1.1 at $x_0 = \pm 3$ for (a)$\eta = 15$. (b) $\eta = 100$.

We note that in Example 2.1 the lower CCT converges to the original function as η tends to positive ∞ and when we used the translation-invariant to solve this example we find that it has no effect in the values of the lower CCT.

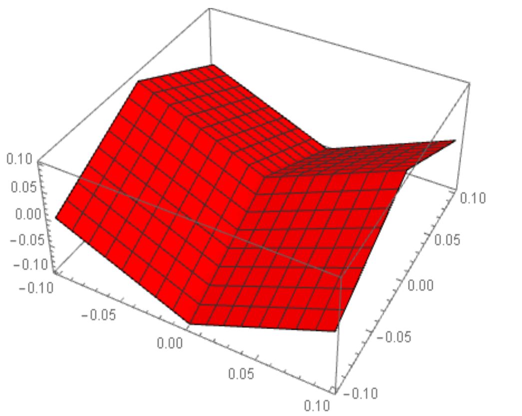
The following example gives a function of two variable and we need to solve the singularity of this function.

Example 2.2. Consider $f(x,y) = |x| - |y|$. Then for $\eta \geq 0$, by using Algorithm 2.1 the lower and upper CCT are given by.

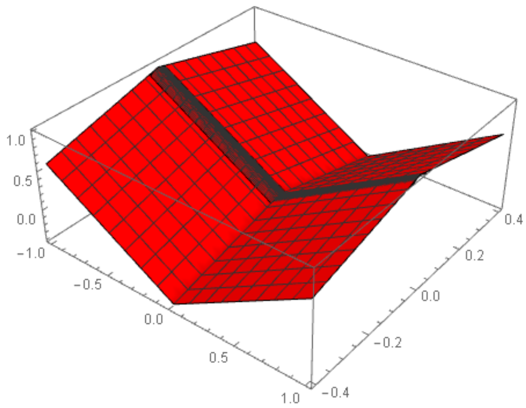
$$C_\eta^l(f)(x,y) = \begin{cases} x - \frac{1}{4\eta} - \eta y^2 & \text{if } x \geq 0, |y| \leq \frac{1}{2\eta}, \\ -x - \frac{1}{4\eta} - \eta y^2 & \text{if } x \leq 0, |y| \leq \frac{1}{2\eta}, \end{cases}$$

$$C_\eta^u(f)(x,y) = \begin{cases} y + \frac{1}{4\eta} + \eta x^2 & \text{if } y \leq 0, |x| \leq \frac{1}{2\eta}, \\ -y + \frac{1}{4\eta} + \eta x^2 & \text{if } y \geq 0, |x| \leq \frac{1}{2\eta}. \end{cases}$$

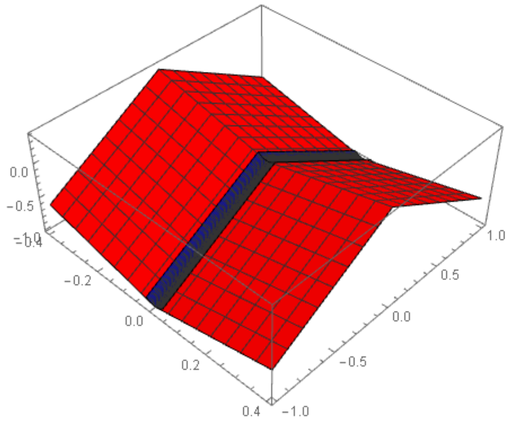
Fig 2, displays the graphs of these transforms with $\eta = 60$.



(a)



(b)



(c)

Fig 2: (a) The function $f(x,y) = |x| - |y|$. (b) The lower CCT for Example 2.2 with $\eta = 60$. (c) The upper CCT for Example 2.2 with $\eta = 60$.

We note that in Example 2.2 the lower CCT below the function and the upper CCT above the function.

We recall definition of Hölder and Lipschitz continuous functions, see [3].

Definition 2.6. Consider the function

$f : \mathbb{R}^n \to \mathbb{R}$, we say that f is Hölder continuous function if there exists a positive constant $L > 0$ such that

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq L\,|\mathbf{x} - \mathbf{y}|^{\alpha}, \quad (2.8)$$

with $0 < \alpha < 1$. When $\alpha = 1$, we say that f is Lipschitz continuous if

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq L\,|\mathbf{x} - \mathbf{y}|. \quad (2.9)$$

Following [15], we recall some more important results related to CCT of Lipschitz and Hölder function.

Theorem 2.1. Consider the function

$f : \mathbb{R}^n \to \mathbb{R}$ be a Hölder function with $0 < \alpha < 1$, then for $\eta > 0$ and $L > 0$, we have

$$C_\eta^l(f)(\mathbf{x_0}) \leq f(\mathbf{x_0})$$

$$\leq C_\eta^l(f)(\mathbf{x_0}) + L^{\frac{2}{2-\alpha}}\left(\frac{\alpha}{2\eta}\right)^{\frac{\alpha}{2-\alpha}}\left(1 - \frac{\alpha}{2}\right),$$

for f is Lipschitz function with Lipschitz constant $L > 0$, we have

$$C_\eta^l(f)(\mathbf{x}) \leq f(\mathbf{x}) \leq C_\eta^l(f)(\mathbf{x}) + \frac{L^2}{4\eta} \quad (2.10)$$

Now, we mention the localization version of the definition of the convex envelope for a given function, see [15].

Definition 1.7. Let $r > 0$ and $\mathbf{x_0} \in \mathbb{R}^n$.

$B(\mathbf{x_0}, r)$ is called the open ball of center $\mathbf{x_0}$ and radius r, $\bar{B}(\mathbf{x_0}, r)$ is the corresponding closed ball. Let $f : \bar{B}(\mathbf{x}_0, r) \to \mathbb{R}$ is a bounded function then the local convex envelope of f at x_0 is defined by

$$\mathbf{CO}_{\bar{B}(\mathbf{x}_0,r)}[f](\mathbf{x}_0) = \inf_{1 \leq i \leq n+1} \left\{\sum_{i=1}^{n+1} \eta_i f(\mathbf{x}_i) : \sum_{i=1}^{n+1} \eta_i = 1, \sum_{i=1}^{n+1} \eta_i \mathbf{x}_i = \mathbf{x}_0, \eta_i \geq 0, |\mathbf{x}_i - \mathbf{x}_0| \leq r, \mathbf{x}_i \in \mathbb{R}^n\right\}. \quad (2.11)$$

We recall the locality properties of CCT for bounded and Lipschitz continuous functions presented in [11,15]. Throughout this paper, we call $\mathbf{R}_\eta$ as locality radius.

Theorem 2.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ bounded function such that for positive constant M, we have $|f(x)| \leq M$, then the locality radius of bounded function is given by $\mathbf{R}_\eta = 2\sqrt{\frac{2M}{\eta}}$ for $\eta > 0$.

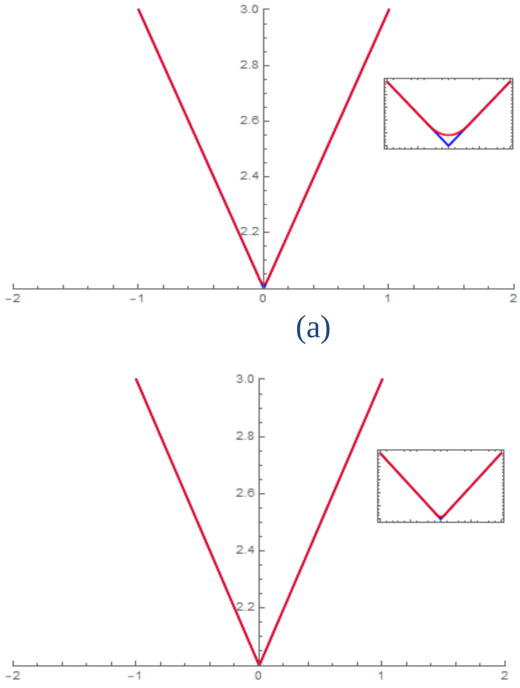
Theorem 2.3. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous function with Lipschitz constant $L > 0$, then the locality radius of Lipschitz function is given by $\mathbf{R}_\eta = \frac{(2+\sqrt{2})L}{\eta}$ for all $\eta > 0$ and for every $\mathbf{x} \in \mathbb{R}^n$.

We will give example of Lipschitz function to show Theorem 2.3.

Example 2.3. Consider $f(x) = |x| + \gamma, x \in \mathbb{R}, \gamma \geq 0$ is Lipschitz function, since $f(x)$ satisfies condition (2.3) of the upper CCT so according to Algorithm 2.1 for $\eta > 0$ the upper CCT is given by

$$C_\eta^u(f)(\mathrm{x}) = \begin{cases} \frac{1}{4\eta} + \gamma + \eta(x)^2 & \text{if } |x| \leq \frac{1}{2\eta} \\ |x| + \gamma & \text{if } |x| \geq \frac{1}{2\eta} \end{cases}$$

The graph of this example is shown in Figure 3.



(a)

(b)

Fig 3: The upper CCT for Example 2.3 for (a)$\eta = 15$. (b)$\eta = 70$.

Since $f(x) = |x| + \gamma,\ x \in \mathbb{R}, \gamma \geq 0$ is Lipschitz function with $L = 1$ which satisfies (2.9). The locality radius of Example 2.3 equal $\frac{1}{2\eta}$ which is less than the locality radius of Theorem 2.3. So according to Theorem 2.3 when the function is Lipschitz then the value of the upper CCT depends on the values of the function f in a closed ball with radius $\frac{1}{2\eta}$ and the locality radius will decreasing when η increasing and tends to infinity.

3. On the locality of CCT

In this section, we present the locality property of the CCT for Hölder function. Firstly, we shall introduce some results that will help us to construct our main results in this section.

Lemma 3.1. Assume that $z > 0$ such that $z^{\beta} > az + b$ with a, b are positive constants and $\beta > 1$. Then for any $z_* > 0$ with $z_*^{\beta} < a z_* + b$, we have $z_* < z$.

Proof. The proof key is based on the contradiction. Assume that $z > 0$ such that $z^{\beta} > az + b$ and there exists $z_* > 0$ with $z_*^{\beta} < a z_* + b$ such that $z_* > z$. Since if $z_* > z$ then $z_*^{\beta} > z^{\beta}$ hence $a z_* + b > z_*^{\beta} > z^{\beta}$ which contradict the assumption $z^{\beta} > az + b$ so that $z_* < z$. Now we need to solve the inequality $z_*^{\beta} < a z_* + b$. The solution of the inequality is equivalent to find the upper bound of the solution of the inequality $z^{\beta} > az + b$. Since $z^{\beta} > az + b,\ z > 0$. Take $z = \frac{b}{a} u$, $u > 0$. Therefore, $\frac{u^{\beta}}{u+1} > b\left(\frac{a}{b}\right)^{\beta}$ equivalent to $\frac{u^{\beta-1}}{\frac{1}{u}+1} > b\left(\frac{a}{b}\right)^{\beta}$.

Take $u^{\beta-1} = 2b\left(\frac{a}{b}\right)^{\beta}$, then

$u = \left(2b\left(\frac{a}{b}\right)^{\beta}\right)^{\frac{1}{\beta-1}}$. Therefore,

$$\frac{2b\left(\frac{a}{b}\right)^{\beta}}{1+\frac{1}{\left(2b\left(\frac{a}{b}\right)^{\beta}\right)^{\frac{1}{\beta-1}}}} > b\left(\frac{a}{b}\right)^{\beta}. \quad (3.1)$$

Dividing both side of (3.1) by $\left(b\left(\frac{a}{b}\right)^{\beta}\right)$, gives $\frac{2}{1+\frac{1}{\left(2b\left(\frac{a}{b}\right)^{\beta}\right)^{\frac{1}{\beta-1}}}} > 1$.

So that, we have

$$2 > 1 + \frac{1}{\left(2b\left(\frac{a}{b}\right)^{\beta}\right)^{\frac{1}{\beta-1}}}, \quad (3.2)$$

(3.2) holds by taking condition $\left(2b\left(\frac{a}{b}\right)^{\beta}\right)^{\frac{1}{\beta-1}} > 1$. So that,

$$z_* < \frac{b}{a}\left(2b\left(\frac{a}{b}\right)^{\beta}\right)^{\frac{1}{\beta-1}} \quad \text{such that} \quad \left(2b\left(\frac{a}{b}\right)^{\beta}\right)^{\frac{1}{\beta-1}} > 1. \quad (3.3)$$

(3.3) equivalent to

$$z_* < (2a)^{\frac{1}{\beta-1}} \text{ such that } (2a)^{\frac{1}{\beta-1}}\frac{a}{b} > 1. \quad (3.4)$$

Then there exists $z = \frac{b}{a}\left(2b\left(\frac{a}{b}\right)^{\beta}\right)^{\frac{1}{\beta-1}}$ such that $z_* < z$.

To improve the previous result we will give the next corollary.

Corollary 3.1. Optimization the solution $z = (2a)^{\frac{1}{\beta-1}}$ in the previous Lemma 3.1.

Proof. We will use the Newton Raphson method to improve $z = (2a)^{\frac{1}{\beta-1}}$.

Let $f(z) = z^{\beta} - az - b$, then $f'(z) = \beta z^{\beta-1} - a$. Since

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, n = 0,1,2,\ldots$$

$$= \frac{(\beta-1)z_n^{\beta} + b}{\beta z_n^{\beta-1} - a}$$

at $n = 0$, $z_0 = (2a)^{\frac{1}{\beta-1}}$, we have

$$z_1 = \frac{(\beta-1)(2a)^{\frac{\beta}{\beta-1}} + b}{a(2\beta-1)}.$$

Therefore,

$$z_* < \frac{(\beta-1)(2a)^{\frac{\beta}{\beta-1}} + b}{a(2\beta-1)} < z_0, \quad \text{such that} \quad (2a)^{\frac{1}{\beta-1}}\frac{a}{b} > 1. \quad (3.5)$$

Now, we compute the locality radius of the CCT of the Hölder functions.

Theorem 3.1. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a Hölder function with constant $L > 0$, $\eta > 0$ and $\mathbf{x} \in \mathbb{R}^n$, then the locality radius of $C_{\eta}^{u}(\cdot)$ and $C_{\eta}^{l}(\cdot)$ is given by

$$\mathbf{R}_{\eta} = 2^{\frac{5-2\alpha}{2-\alpha}}\left(\frac{L}{\eta}\right)^{\frac{1}{2-\alpha}}.$$

Proof. Consider that $\phi(\mathbf{x}) = f(\mathbf{x}) + \eta\,|\mathbf{x}|^2$ and without loss of generality we assume that $\mathbf{x}_0 = 0$ and by using (2.7) then there exist $(\eta_i, \mathbf{x}_i) \in \mathbb{R} \times \mathbb{R}^n$ for $i = 1, \ldots, n+1$ such that

$$\mathbf{CO}[\phi](\mathbf{0}) = \{\textstyle\sum_{i=1}^{n+1} \eta_i \phi(\mathbf{x}_i) : \sum_{i=1}^{n+1} \eta_i = 1, \sum_{i=1}^{n+1} \eta_i \mathbf{x}_i = \mathbf{0},\ \eta_i \geq 0, \mathbf{x}_i \in \mathbb{R}^n\}. \quad (3.6)$$

Also, according to Definition 2.4, there exists an affine function

$$\ell(\mathbf{x}) = <\mathbf{a}^*, \mathbf{x}> + b^*,\ \mathbf{a}^* \in \mathbb{R}^n, b^* \in \mathbb{R}$$

Such that

$$\ell(\mathbf{x}) \leq \phi(\mathbf{x})\ \forall \mathbf{x} \in \mathbb{R}^n, \quad (3.7)$$

$$\ell(\mathbf{x}_i) = \phi(\mathbf{x}_i)\ \forall i = 1,2,\ldots,n+1. \quad (3.8)$$

By taking convex envelope to both side in (3.8), we have

$\mathbf{CO}[\ell(\mathbf{x}_i)] = \ell(\mathbf{x}_i) = \mathbf{CO}[\phi(\mathbf{x}_i)]$. Following (3.6), we conclude that

$$\begin{aligned}\mathbf{CO}[\phi](\mathbf{0}) &= \textstyle\sum_{i=1}^{n+1} \eta_i \ell(\mathbf{x}_i) = \sum_{i=1}^{n+1} \eta_i(\mathbf{a}^* \cdot \mathbf{x}_i + b^*) \\ &= \textstyle\sum_{i=1}^{n+1} \mathbf{a}^* \cdot (\eta_i \mathbf{x}_i) + \sum_{i=1}^{n+1} \eta_i b^* \\ &= \mathbf{a}^* \cdot \textstyle\sum_{i=1}^{n+1} \eta_i \mathbf{x}_i + b^* \sum_{i=1}^{n+1} \eta_i = b^*.\end{aligned}$$

Moreover, from the definition of $\ell(\mathbf{x})$ we have $\ell(0) = b^*$, thus

$\ell(0) = b^* = \mathbf{CO}[\phi](0)$.

To get an estimate on $|\mathbf{a}^*|$, assume that $\mathbf{x} = \frac{\mathbf{a}^*}{2\eta}$ in (3.7), we obtain

$$\mathbf{a}^* \cdot \left(\frac{\mathbf{a}^*}{2\eta}\right) + b^* \le f\left(\frac{\mathbf{a}^*}{2\eta}\right) + \eta \left|\frac{\mathbf{a}^*}{2\eta}\right|^2,$$

which is equivalent to

$$\frac{|\mathbf{a}^*|^2}{4\eta} \le f\left(\frac{\mathbf{a}^*}{2\eta}\right) - b^*. \qquad (3.9)$$

Also, we can rewrite (3.9) as follows,

$$\frac{|\mathbf{a}^*|^2}{4\eta} \le f\left(\frac{\mathbf{a}^*}{2\eta}\right) - f(0) + f(0) - b^*. \qquad (3.10)$$

Following Theorem 3.1 at $\mathbf{x} = 0$, we have

$$f(0) - C_\eta^l(f)(0) \le L^{\frac{2}{2-\alpha}}\left(\frac{\alpha}{2\eta}\right)^{\frac{\alpha}{2-\alpha}}\left(1 - \frac{\alpha}{2}\right)$$

and since f is Hölder function, then we deduce from (3.10) that

$$\left(\frac{|\mathbf{a}^*|}{2}\right)^2 \le L(\eta)^{1-\alpha}\left(\frac{|\mathbf{a}^*|}{2}\right)^\alpha + L^{\frac{2}{2-\alpha}}\left(\frac{\alpha}{2}\right)^{\frac{\alpha}{2-\alpha}}(\eta)^{\frac{2(1-\alpha)}{2-\alpha}}\left(1 - \frac{\alpha}{2}\right).$$

Take

$\mathbf{w} = \left(\frac{|\mathbf{a}^*|}{2}\right)^\alpha$, $c_0 = L(\eta)^{1-\alpha}$,

$c_1 = L^{\frac{2}{2-\alpha}}\left(\frac{\alpha}{2}\right)^{\frac{\alpha}{2-\alpha}}(\eta)^{\frac{2(1-\alpha)}{2-\alpha}}\left(1 - \frac{\alpha}{2}\right)$ and $c = \frac{2}{\alpha}$. Then, we have $\mathbf{w}^c \le c_0\mathbf{w} + c_1$ such that $c_0 > 0$, $c_1 > 0$ and $c > 1$.

By applying ondition (3.3) of Lemma 3.1

$$(2c_0)^{\frac{1}{c-1}}\frac{c_0}{c_1} = (2L(\eta)^{1-\alpha})^{\frac{\alpha}{2-\alpha}} \cdot \left(\frac{L(\eta)^{1-\alpha}}{L^{\frac{2}{2-\alpha}}\left(\frac{\alpha}{2}\right)^{\frac{\alpha}{2-\alpha}}(\eta)^{\frac{2(1-\alpha)}{2-\alpha}}\left(1 - \frac{\alpha}{2}\right)}\right) = (2)^{\frac{(\alpha+2)}{2-\alpha}}\frac{(\alpha)^{\frac{(-\alpha)}{2-\alpha}}}{2-\alpha} > 1.$$

Since $(0 < \alpha < 1)$ so that the condition holds. Therefore, there exists an upper bound of $\mathbf{w}$

$$\mathbf{w} \le (2c_0)^{\frac{1}{c-1}} = (2L(\eta)^{1-\alpha})^{\frac{\alpha}{2-\alpha}}.$$

Since $|\mathbf{a}^*| = 2\mathbf{w}^{\frac{1}{\alpha}}$ hence

$$|\mathbf{a}^*| \le 2(2L(\eta)^{1-\alpha})^{\frac{1}{2-\alpha}} \qquad (3.11)$$

Now, we seek to estimate the radius of locality that is $\mathbf{R}_{\eta,L,\alpha}$ from (3.8), we have

$$\begin{aligned}\eta|\mathbf{x}_i|^2 &= \mathrm{b}^* - f(\mathbf{x}_i) + \mathbf{a}^* \cdot \mathbf{x}_i\\ &= \mathrm{b}^* - f(0) + f(0) - f(\mathbf{x}_i) + \mathbf{a}^* \cdot \mathbf{x}_i\\ &\le L|\mathbf{x}_i|^\alpha + |\mathbf{a}^*| \cdot \mathbf{x}_i.\end{aligned}$$

Therefore,

$$\eta|\mathbf{x}_i|^2 \le L|\mathbf{x}_i|^\alpha + |\mathbf{a}^*| \cdot |\mathbf{x}_i|. \qquad (3.12)$$

Take $\mathbf{u} = |\mathbf{x}_i|^\alpha$, then $|\mathbf{x}_i| = \mathbf{u}^{\frac{1}{\alpha}}$ so that (2.12) equivalent to

$$\eta\mathbf{u}^{\frac{2}{\alpha}} \le L\mathbf{u} + |\mathbf{a}^*| \cdot \mathbf{u}^{\frac{1}{\alpha}}.$$

$$\mathbf{u}^{\frac{2-\alpha}{\alpha}} \le \frac{L}{\eta} + \frac{|\mathbf{a}^*|}{\eta} \cdot \mathbf{u}^{\frac{1-\alpha}{\alpha}}.$$

Take $\mathbf{v} = \mathbf{u}^{\frac{1-\alpha}{\alpha}}$, then $\mathbf{u} = \mathbf{v}^{\frac{\alpha}{1-\alpha}}$

$$\mathbf{v}^{\frac{2-\alpha}{1-\alpha}} \le \frac{L}{\eta} + \frac{|\mathbf{a}^*|}{\eta} \cdot \mathbf{v}$$

Let $\gamma = \frac{2-\alpha}{1-\alpha}$, $\mathbf{c}_2 = \frac{|\mathbf{a}^*|}{\eta}$ and $c_3 = \frac{L}{\eta}$ then, we have $\mathbf{v}^\gamma \le \frac{|\mathbf{a}^*|}{\eta} \cdot \mathbf{c}_2 + c_3$ such that $\mathbf{c}_2 > 0$, $c_3 > 0$ and $\gamma > 1$. By applying condition (3.3) of Lemma 3.1

$$(2\boldsymbol{c}_2)^{\frac{1}{\gamma-1}}\frac{\boldsymbol{c}_2}{c_3} = 2^{1-\alpha}\frac{c_2^{2-\alpha}}{c_3} = \frac{(2|\mathbf{a}^*|)^{2-\alpha}}{2L\eta^{(1-\alpha)}} > 1$$

Therefore, $\mathbf{v}$ has an upper bound, $\mathbf{v} \le (2\boldsymbol{c}_2)^{\frac{1}{\gamma-1}} = (2\boldsymbol{c}_2)^{1-\alpha}$ and $|\mathbf{x}_i| = \mathbf{v}^{\frac{1}{1-\alpha}}$. So $|\mathbf{x}_i| \le 2\mathbf{c}_2 = 2\frac{|\mathbf{a}^*|}{\eta}$.

If we using the result of $|\mathbf{a}^*|$ from (3.11), we have

$$|\mathbf{x}_i| \le 2^{\frac{5-2\alpha}{2-\alpha}}\left(\frac{L}{\eta}\right)^{\frac{1}{2-\alpha}}. \qquad (3.13)$$

Corollary 3.2. Optimization the radius of locality of the previous Theorem 3.1

Proof. Condition (3.5) in Corollary 3.1 satisfies. Therefore, we have

$\mathbf{w} < \frac{(\beta-1)(2c_0)^{\frac{\beta}{\beta-1}} + c_1}{c_0(2\beta-1)}$. So

$$\mathbf{w} \le \left(\frac{2-\alpha}{4-\alpha}\right)2^{\frac{2}{2-\alpha}}L^{\frac{\alpha}{2-\alpha}}\eta^{\frac{\alpha(1-\alpha)}{2-\alpha}} + 2^{\frac{-\alpha}{2-\alpha}}\frac{\alpha^{\frac{2}{2-\alpha}}}{(4-\alpha)}L^{\frac{\alpha}{2-\alpha}}\eta^{1-\alpha}.$$

Therefore,

$$|\mathbf{a}^*| \le 2\left(\left(\frac{2-\alpha}{4-\alpha}\right)2^{\frac{2}{2-\alpha}}L^{\frac{\alpha}{2-\alpha}}\eta^{\frac{\alpha(1-\alpha)}{2-\alpha}} + 2^{\frac{-\alpha}{2-\alpha}}\frac{\alpha^{\frac{2}{2-\alpha}}}{(4-\alpha)}L^{\frac{\alpha}{2-\alpha}}\eta^{1-\alpha}\right)^{\frac{1}{\alpha}}. \qquad (3.14)$$

Also if we using the result of $|\mathbf{a}^*|$ from (3.14). Then, we have

$$|\mathbf{x}_i| = 2\mathbf{c}_2 = 2\frac{|\mathbf{a}^*|}{\eta}$$

$$\le \frac{4}{\eta}\left(\left(\frac{2-\alpha}{4-\alpha}\right)2^{\frac{2}{2-\alpha}}L^{\frac{\alpha}{2-\alpha}}\eta^{\frac{\alpha(1-\alpha)}{2-\alpha}} + 2^{\frac{-\alpha}{2-\alpha}}\frac{\alpha^{\frac{2}{2-\alpha}}}{(4-\alpha)}L^{\frac{\alpha}{2-\alpha}}\eta^{1-\alpha}\right)^{\frac{1}{\alpha}}. \qquad (3.15)$$

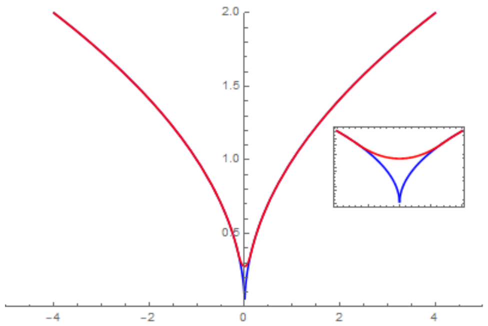
for $i = 1, 2, \ldots, n + 1$. This completes the proof.

We will give an example of Hölder function to show the previous Theorem 3.1.

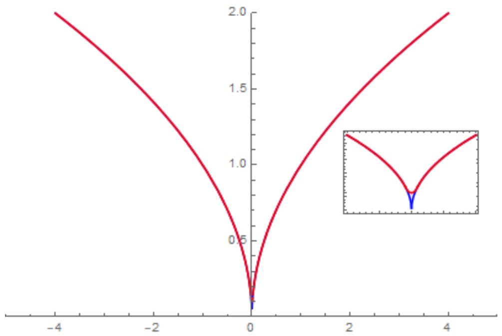
Example 3.1. Consider $f(x) = \sqrt{|x|}, x \in \mathbb{R}$. Since $f(x)$ satisfies condition (2.3) of the upper CCT, so according to Algorithm 2.1 for $\eta > 0$ the upper CCT is given by

$$C_\eta^u(f)(x) = \begin{cases} \frac{3}{4}\left(\frac{1}{4\eta}\right)^{\frac{1}{3}} + \eta(|x|)^2 & if\, |x| \le \left(\frac{1}{4\eta}\right)^{\frac{2}{3}} \\ \sqrt{|x|} & if\, |x| \ge \left(\frac{1}{4\eta}\right)^{\frac{2}{3}} \end{cases}$$

The graph of this example is shown in Figure 4.



(a)



(b)

Fig 4: (a) The upper CCT for Example 3.1 for (a)$\eta = 5$, (b)$\eta = 100$.

Since $f(x) = \sqrt{|x|}, x \in \mathbb{R}$ is Hölder function which satisfies (2.8) with $L = 1$, $\alpha = \frac{1}{2}$. We note that the locality property for Hölder function holds and the value of the upper CCT depend only on the values of the function f in a closed ball with radius $\left(\frac{1}{4\eta}\right)^{\frac{2}{3}}$ and the radius of locality of Example 3.1 less or equal to $\left(\frac{1}{4\eta}\right)^{\frac{2}{3}}$ and this equivalent to the radius of locality in the previous Theorem 3.1 in (3.13) so the radius of locality in (3.13) is better than the other radius in (3.15).

4. Hausdorff Stability of the n-CCT.

In this section, we will study the Hausdorff stability for the n-upper CCT, similarly (lower) by showing that the n-upper CCT is Hausdorff-Lipschitz continuous with respect to closed sample sets at Hölder function f. The Hausdorff stability results for bounded uniformly continuous positive function with respect to closed sample sets has been presented in [12]. Hausdorff stability is established in [1], when the functions be a bounded with respect to the Hausdorff distance between graphs of the sampled functions.

We give the next result of the n-th root of Hölder function, that help us later to prove Hausdorff stability with respect to closed sample sets.

Lemma 4.1. The n-th root of Hölder function is Hölder as well.

Proof. Without loss of generality, we suppose that $0 \le \mathbf{y} \le \mathbf{x}$. Since n-th root function is increasing function, hence we have $g = \sqrt[n]{f(\mathbf{x})} - \sqrt[n]{f(\mathbf{y})}$ such that $g \ge 0$. So that

$f(\mathbf{x}) - f(\mathbf{y}) = \left(\sqrt[n]{f(\mathbf{x})}\right)^n - f(\mathbf{y})$

$= \left(\sqrt[n]{f(\mathbf{x})} + g\right)^n - f(\mathbf{y})$ (By using Binomial Theorem)

$$= \left(\left(\sqrt[n]{f(\mathbf{x})}\right)^n + \cdots + (g)^n\right) - f(\mathbf{y}) \ge g^n.$$

Hence, we have

$$g = \sqrt[n]{f(\mathbf{x})} - \sqrt[n]{f(\mathbf{y})} \le \sqrt[n]{f(\mathbf{x}) - f(\mathbf{y})},$$

since f is Hölder function so satisfies the fact in (2.8). Therefore, we have

$$\sqrt[n]{f(\mathbf{x})} - \sqrt[n]{f(\mathbf{y})} \le L^{\frac{1}{n}} (\mathbf{x} - \mathbf{y})^{\frac{\alpha}{n}}. \quad (4.1)$$

This complete the proof.

We shall introduce some definitions and results that will help us to construct our main result in this section, see [12].

Definition 4.1. For the function

$f : \mathbb{R}^m \to \mathbb{R}$ is positive bounded such that

$0 < f(\mathbf{x}) < M$ for all $\mathbf{x} \in \mathbb{R}^m$ and E is a closed subset of $\mathbb{R}^m$, we define the distance-like function ϖ as follow:-

$$\varpi_{(n,\eta,f)}(\mathbf{x}, E) = \inf_{\mathbf{y} \in E} \left\{ |\mathbf{y} - \mathbf{x}| - \sqrt[n]{\frac{f(\mathbf{y})}{\eta}} \right\}, \qquad \mathbf{x} \in \mathbb{R}^m$$

$$D_{(n,\eta,f)}(\mathbf{x}, E) = -\sqrt[n]{\eta}\, \min\{0, \varpi_{(n,\eta,f)}(\mathbf{x}, E)\}.$$

By using [2] we give the definition of Hausdorff distance between two sets.

Definition 4.2. Consider that E, K two non empty closed and bounded subsets of $\mathbb{R}^m$. The

Hausdorff distance between E and K is defined as follow:

$dist_{\mathcal{H}}(E,K) = inf\{\delta > 0 : E \subset K^{\delta} \text{ and } K \subset E^{\delta}\}.$

This definition equivalent to

$$\mathrm{dist}_{\mathcal{H}}(E,K) = \max\left\{\sup_{\mathrm{x}\in \mathrm{E}} \varpi(\mathbf{x},E), \sup_{\mathrm{x}\in \mathrm{E}} \varpi(\mathbf{x},k)\right\}.$$

We will give a Hausdorff continuity of distance-like function ϖ.

Corollary 4.1. Suppose that $f : \mathbb{R}^m \to \mathbb{R}$ is Hölder and positive bounded function. For any two non-empty closed subsets E, K of $\mathbb{R}^m$ such that $\mathrm{dist}_{\mathcal{H}}(E,K) < +\infty$. Then

$$\left|\varpi_{(n,\eta,f)}(\mathbf{x},E) - \varpi_{(n,\eta,f)}(\mathbf{x},K)\right| \leq \sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}\sqrt{\mathrm{dist}_{\mathcal{H}}^{2}(E,K) + \mathrm{dist}_{\mathcal{H}}^{\frac{2\alpha}{n}}(E,K)}$$

for all $\mathbf{x} \in \mathbb{R}^m$.

Proof. Since the function f is Hölder, hence there exist $\mathbf{x}^K \in K$ such that $\varpi_{(n,\eta,f)}(\mathbf{x},K) = |\mathbf{x}^K - \mathbf{x}| - \sqrt[n]{\frac{f(\mathbf{x}^K)}{\eta}}$, by using Definition 3.2 of distance Hausdorff between two sets for $\mathbf{x}^K \in K$ there exist $\mathbf{x}^E \in E$ such that $|\mathbf{x}^E - \mathbf{x}^K| < \delta$. So that

$$\begin{aligned}
&\varpi_{(n,\eta,f)}(\mathbf{x},E) - \varpi_{(n,\eta,f)}(\mathbf{x},K)\\
&= |\mathbf{x}^E - \mathbf{x}| - \sqrt[n]{\frac{f(\mathbf{x}^E)}{\eta}} - |\mathbf{x}^K - \mathbf{x}| + \sqrt[n]{\frac{f(\mathbf{x}^K)}{\eta}}\\
&= |\mathbf{x}^E - \mathbf{x}| - |\mathbf{x}^K - \mathbf{x}| - \left(\sqrt[n]{\frac{f(\mathbf{x}^E)}{\eta}} - \sqrt[n]{\frac{f(\mathbf{x}^K)}{\eta}}\right)\\
&\leq |\mathbf{x}^E - \mathbf{x}^K| + \frac{1}{\sqrt[n]{\eta}}\left|\sqrt[n]{f(\mathbf{x}^E)} - \sqrt[n]{f(\mathbf{x}^K)}\right|
\end{aligned}$$

By using Lemma 4.1, we have

$$\varpi_{(n,\eta,f)}(\mathbf{x},E) - \varpi_{(n,\eta,f)}(\mathbf{x},K) \leq |\mathbf{x}^E - \mathbf{x}^K| + \sqrt[n]{\frac{L}{\eta}}|\mathbf{x}^E - \mathbf{x}^K|^{\frac{\alpha}{n}}. \qquad (4.3)$$

By using Cauchy-Schwarz inequality in the right side, so (4.3) equivalent to

$$\varpi_{(n,\eta,f)}(\mathbf{x},E) - \varpi_{(n,\eta,f)}(\mathbf{x},K) \leq \sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}\sqrt{\delta^2 + \delta^{\frac{2\alpha}{n}}} \quad \text{for all} \quad \delta > \mathrm{dist}_{\mathcal{H}}(\mathrm{E},\mathrm{K}).$$

Therefore

$$\varpi_{(n,\eta,f)}(\mathbf{x},E) - \varpi_{(n,\eta,f)}(\mathbf{x},K) \leq \sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}\sqrt{\mathrm{dist}_{\mathcal{H}}(\mathrm{E},\mathrm{K})^2 + \mathrm{dist}_{\mathcal{H}}(\mathrm{E},\mathrm{K})^{\frac{2\alpha}{n}}}$$

Similarly, we can conclude that

$$\varpi_{(n,\eta,f)}(\mathbf{x},K) - \varpi_{(n,\eta,f)}(\mathbf{x},E) \leq \sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}\sqrt{\mathrm{dist}_{\mathcal{H}}^{2}(E,K) + \mathrm{dist}_{\mathcal{H}}^{\frac{2a}{n}}(E,K)}$$

This completes the proof.

We will give a Hausdorff continuity of $D^n_{(n,\eta,f)}(\mathbf{x},E)$.

Corollary 4.2. Suppose that $f : \mathbb{R}^m \to \mathbb{R}$ is a Hölder and positive bounded function. For any two non-empty closed subsets E, K of $\mathbb{R}^m$. Then, we have

$$\left|D^n_{(n,\eta,f)}(\mathbf{x},E) - D^n_{(n,\eta,f)}(\mathbf{x},K)\right| \leq n(\sqrt[n]{M})^{n-1}\sqrt[n]{\eta}\sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}\sqrt{\mathrm{dist}_{\mathcal{H}}^{2}(E,K) + \mathrm{dist}_{\mathcal{H}}^{\frac{2a}{n}}(E,K)}$$

Proof. Since

$$\left|D^n_{(n,\eta,f)}(\mathbf{x},E) - D^n_{(n,\eta,f)}(\mathbf{x},K)\right| \leq \left|D_{(n,\eta,f)}(\mathbf{x},E) - D_{(n,\eta,f)}(\mathbf{x},K)\right| \cdot \sum_{i=0}^{n-1}\left(D^{n-i-1}_{(n,\eta,f)}(\mathbf{x},E)\cdot D^{i}_{(n,\eta,f)}(\mathbf{x},K)\right). \qquad (4.4)$$

Following (4.2) of Definition 4.1, if distance-like function $\varpi_{(n,\eta,f)}(\mathbf{x},E) \geq 0$ then $D_{(n,\eta,f)}(\mathbf{x},E) = 0$ and if $\varpi_{(n,\eta,f)}(\mathbf{x},E) < 0$ then $\varpi_{(n,\eta,f)}(\mathbf{x},E) = |\mathbf{x}^{\mathrm{E}} - \mathbf{x}| - \sqrt[\mathrm{n}]{\frac{f(\mathbf{x}^{\mathrm{E}})}{\eta}} < 0$ for $\mathbf{x}^{\mathrm{E}} \in \mathrm{E}$. Therefore,

$$\begin{aligned}
D_{(n,\eta,f)}(\mathbf{x},E) &= -\sqrt[n]{\eta}\left(|\mathbf{x}^E - \mathbf{x}| - \sqrt[n]{\frac{f(\mathbf{x}^E)}{\eta}}\right)\\
&= \sqrt[n]{\eta}\left(\sqrt[n]{\frac{f(\mathbf{x}^E)}{\eta}} - |\mathbf{x}^E - \mathbf{x}|\right)\\
&\leq \sqrt[n]{\eta}\cdot\sqrt[n]{\frac{f(\mathbf{x}^E)}{\eta}} \leq \sqrt[n]{f(\mathbf{x}^E)}
\end{aligned}$$

Since the function f is bounded then, we have $\mathrm{D}_{(n,\eta,f)}(\mathbf{x},\mathrm{E}) \leq \sqrt[\mathrm{n}]{\mathrm{M}}$ and M is positive constant. Therefore, we conclude that

$$\begin{aligned}
&\sum_{i=0}^{n-1}\left(D^{n-i-1}_{(n,\eta,f)}(\mathbf{x},E)\cdot D^{i}_{(n,\eta,f)}(\mathbf{x},K)\right)\\
&\leq \sum_{i=0}^{n-1}(\sqrt[n]{M})^{n-i-1}\cdot(\sqrt[n]{M})^{i}\\
&= \sum_{i=0}^{n-1}(\sqrt[n]{M})^{n-1} = n(\sqrt[n]{M})^{n-1}
\end{aligned}$$

(4.5)

$$\left|D_{(n,\eta,f)}(\mathbf{x},E) - D_{(n,\eta),f)}(\mathbf{x},K)\right| = \left| -\sqrt[n]{\eta}\, min\left(0, \varpi_{(n,\eta,f)}(\mathbf{x},E)\right) + \right.$$

$\sqrt[n]{\eta}\, min\left(0, \varpi_{(n,\eta,f)}(\mathbf{x}, K)\right)|.$

By using the relation

$min\,(a,b) = \frac{(a+b)-|a-b|}{2}$, we have

$$\left|D_{(n,\eta,f)}(\mathbf{x},E) - D_{(n,\eta,f)}(\mathbf{x},K)\right|$$
$$= \frac{\sqrt[n]{\eta}}{2}\,|\varpi_{(n,\eta,f)}(\mathbf{x},E) - \left|\varpi_{(n,\eta,f)}(\mathbf{x},E)\right|$$
$$-\left(\varpi_{(n,\eta,f)}(\mathbf{x},K) - \left|\varpi_{(n,\eta,f)}(\mathbf{x},K)\right|\right)|$$
$$= \frac{\sqrt[n]{\eta}}{2}\,|\varpi_{(n,\eta,f)}(\mathbf{x},E) - \varpi_{(n,\eta,f)}(\mathbf{x},K) +$$
$$\left(-\left|\varpi_{(n,\eta,f)}(\mathbf{x},E)\right| + \left|\varpi_{(n,\eta,f)}(\mathbf{x},K)\right|\right)|$$
$$\leq \frac{\sqrt[n]{\eta}}{2}\left(\left|\varpi_{(n,\eta,f)}(\mathbf{x},E) - \varpi_{(n,\eta,f)}(\mathbf{x},K)\right|\right.$$
$$\left.+\left|-\left(\left|\varpi_{(n,\eta,f)}(\mathbf{x},E)\right| - \left|\varpi_{(n,\eta,f)}(\mathbf{x},K)\right|\right)\right|\right)$$
$$= \sqrt[n]{\eta}\left|\varpi_{(n,\eta,f)}(\mathbf{x},E) - \varpi_{(n,\eta),f)}(\mathbf{x},K)\right|.$$

By using Corollary 4.1, we have

$$\left|D_{(n,\eta,f)}(\mathbf{x},E) - D_{(n,\eta,f)}(\mathbf{x},K)\right|$$
$$\leq \sqrt[n]{\eta}\sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}\sqrt{\operatorname{dist}_{\mathcal{H}}^{2}(E,K) + \operatorname{dist}_{\mathcal{H}}^{\frac{2a}{n}}(E,K)}. \quad (4.6)$$

By compensation from (4.5) and (4.6) in (4.4). This completes the proof.

We need to compute the n-upper CCT of the function $\gamma\chi_{\mathbf{x}_0}$ which help us later in this section.

Lemma 4.2. Let γ is positive constant, $\mathbf{x}_0 \in \mathbb{R}^n$ and E is closed set then for $\eta > 0$, we have

$$C^u_{n,\eta}\left(\gamma\chi_{\mathbf{x}_0}\right)(\mathbf{x}) = \begin{cases} \eta|\mathbf{x}-\mathbf{x}_0|^n + \frac{-\gamma n}{(n-1)}\sqrt[n]{\frac{(n-1)\eta}{\gamma}}|\mathbf{x}-\mathbf{x}_0| + \gamma & \text{if } |\mathbf{x}-\mathbf{x}_0| \leq \sqrt[n]{\frac{\gamma}{(n-1)\eta}}, \\ 0 & \text{if } |\mathbf{x}-\mathbf{x}_0| \geq \sqrt[n]{\frac{\gamma}{(n-1)\eta}}. \end{cases}$$

(4.7)

Proof. By applying Algorithm 2.1 in case the n-upper CCT. We need to calculate the convex envelope of the function

$(-\gamma\chi_{\mathbf{x}_0} + \eta|(\cdot)-\mathbf{x}_0|^n(\mathbf{x})$. Let

$$g(\mathbf{x}) = -\left(\gamma\chi_{\mathbf{x}_0}\right)(\mathbf{x}) + \eta|\mathbf{x}-\mathbf{x}_0|^n = \begin{cases} -\gamma + \eta|\mathbf{x}-\mathbf{x}_0|^n & \text{if } \mathbf{x}_0 \in E, \\ \eta|\mathbf{x}-\mathbf{x}_0|^n & \text{if } \mathbf{x}_0 \notin E. \end{cases}$$

The equation of the tangent line of the curve $\mathbf{y} = g(\mathbf{x}) = -\gamma + \eta|\mathbf{x}-\mathbf{x}_0|^n$ at the point $(0,\gamma)$ is equal to $\mathbf{y} = -\gamma + n\,\eta|\mathbf{x}-\mathbf{x}_0|^n$. We will find the intersection point between the two curves $\mathbf{y} = -\gamma + n\,\eta|\mathbf{x}-\mathbf{x}_0|^n$ and $\mathbf{y} = \eta|\mathbf{x}-\mathbf{x}_0|^n$ so $n\eta|\mathbf{x}-\mathbf{x}_0|^n - \gamma = \mathbf{y} = \eta|\mathbf{x}-\mathbf{x}_0|^n$, hence $|\mathbf{x}-\mathbf{x}_0| = \sqrt[n]{\frac{\gamma}{(n-1)\eta}}$ so the intersection point equal to $\left(\sqrt[n]{\frac{\gamma}{(n-1)\eta}}, \frac{\gamma}{(n-1)}\right)$. Now, we will find the equation of straight line between the two point $(0,-\gamma)$, $\left(\sqrt[n]{\frac{\gamma}{(n-1)\eta}}, \frac{\gamma}{(n-1)}\right)$ so that,

$\mathbf{y} = \frac{\gamma n}{(n-1)}\sqrt[n]{\frac{(n-1)\eta}{\gamma}}|\mathbf{x}-\mathbf{x}_0| - \gamma$. Therefore, we conclude that $C^u_{n,\eta}\left(\gamma\chi_{\mathbf{x}_0}\right)(\mathbf{x})$ is given by (4.7). This complete the proof.

We will give the relation between the n-upper CCT of $f\chi_{\mathrm{E}}(\mathbf{x})$ and $D^n_{(n,\eta,f)}(\cdot,E)\,(\mathbf{x})$.

Corollary 4.3. Suppose that $f : \mathbb{R}^m \to \mathbb{R}$ is Hölder and positive bounded function and for any a non-empty closed subset E of $\mathbb{R}^m$ then for all $\mathbf{x} \in \mathbb{R}^m, \eta > 0$ we have

$$C^u_{n,\eta}(f\chi_{\mathrm{E}})(\mathbf{x}) = C^u_{n,\eta}\left(D^n_{(n,\eta,f)}(\cdot,E)\right)(\mathbf{x}). \quad (4.8)$$

Proof. We want to prove in the first this inequality

$$C^u_{n,\eta}(f\chi_{\mathrm{E}})(\mathbf{x}) \leq C^u_{n,\eta}\left(D^n_{(n,\eta,f)}(\cdot,E)\right)(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^m$. (4.9)

If $\mathbf{x}$ not belong to the set E, so $(f\chi_{\mathrm{E}})(\mathbf{x}) = 0 \leq \left(D^n_{(n,\eta,f)}(\cdot,E)\right)(\mathbf{x})$ and if $\mathbf{x}$ belong to the set E, so

$$\varpi_{(n,\eta,f)}(\mathbf{x},E) = \min_{\mathbf{y}\in E}\left(|\mathbf{y}-\mathbf{x}| - \sqrt[n]{\frac{f(\mathbf{y})}{\eta}}\right).$$

Since $\mathbf{x}$ is the nearest element to itself in a set E. So

$$\varpi_{(n,\eta,f)}(\mathbf{x},E) \leq -\sqrt[n]{\frac{f(\mathbf{x})}{\eta}} < 0$$

So that

$$D_{(n,\eta,f)}(\mathbf{x},E) = -\sqrt[n]{\eta}\varpi_{(n,\eta,f)}(\mathbf{x},E) \geq \sqrt[n]{\eta}\sqrt[n]{\frac{f(\mathbf{x})}{\eta}} = \sqrt[n]{f(\mathbf{x})}.$$

So

$\left(D^n_{(n,\eta,f)}(\cdot,E)\right)(\mathbf{x}) \geq f(\mathbf{x})$. Therefore, we have

$$f(\mathbf{x})\chi_{\mathrm{E}}(\mathbf{x}) \leq \left(D^n_{(n,\eta,f)}(\cdot,E)\right)(\mathbf{x}). \quad (4.10)$$

By applying the ordering property to (4.10), we have

$C^u_{n,\eta}(f\chi_{\mathrm{E}})(\mathbf{x}) \leq C^u_{n,\eta}\left(D^n_{(n,\eta,f)}(\cdot,E)\right)(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m$. (4.11)

We want to prove the inverse of the inequality in (4.9). We have two cases:

The first case: If distance-like function $\varpi_{(n,\eta,f)}(\mathbf{x},E) \geq 0$, then

$\left(D^n_{(n,\eta,f)}(\cdot,E)\right)(\mathbf{x}) = 0$. Therefore, in this case we need to prove that:

$$C^u_{n,\eta}\left(D^n_{(n,\eta,f)}(\cdot,E)\right)(\mathbf{x}) = 0 \leq C^u_{n,\eta}(f\chi_{\mathrm{E}})(\mathbf{x}). \quad (4.12)$$

By using Definition 2.1, we need to prove that the convex envelope of the function

$\left(\eta|\mathbf{u}-\mathbf{x}_0|^n - D^n_{(n,\eta,f)}(\mathbf{u},E)\right)$ equal to zero for all $\mathbf{u}$ in $\mathbb{R}^m$.

Let the affine function $\ell(\mathbf{u}) = 0$ for all $\mathbf{u}$ in $\mathbb{R}^m$and prove that

$$\ell(\mathbf{u}) = 0 = \left[\eta|\mathbf{u}-\mathbf{x}|^n - D^n_{(n,\eta,f)}(\mathbf{u},E)\right] \text{ at } \mathbf{u} = \mathbf{x}, \quad (4.13)$$

$$\ell(\mathbf{u}) = 0 \leq \left[\eta|\mathbf{u}-\mathbf{x}|^n - D^n_{(n,\eta,f)}(\mathbf{u},E)\right] \text{ for all } \mathbf{u} \text{ in } \mathbb{R}^m. \quad (4.14)$$

Since at $\mathbf{u} = \mathbf{x}$, we have

$\left(\eta|\mathbf{u}-\mathbf{x}_0|^n - D^n_{(n,\eta,f)}(\mathbf{u},E)\right) = -D^n_{(n,\eta,f)}(\mathbf{u},E) = 0$ so (4.13) holds.

Now, we prove (4.14) which equivalent to prove that

$$D^n_{(n,\eta,f)}(\mathbf{u},E) \leq \eta|\mathbf{u}-\mathbf{x}_0|^n \text{ for all } \mathbf{u} \text{ in } \mathbb{R}^m. \quad (4.15)$$

If distance-like function $\varpi_{(n,\eta,f)}(\mathbf{u},E) \geq 0$, then $D^n_{(n,\eta,f)}(\mathbf{u},E) = 0$ hence (4.15) holds, and if distance-like function $\varpi_{(n,\eta,f)}(\mathbf{u},E) < 0$, then $D_{(n,\eta,f)}(\mathbf{u},E) = -\sqrt[n]{\eta}\,\varpi_{(n,\eta,f)}(\mathbf{u},E)$. So

$$D^n_{(n,\eta,f)}(\mathbf{u},E) = \left[-\sqrt[n]{\eta}\cdot \min_{\mathbf{y}\in E}\left(|\mathbf{y}-\mathbf{u}| - \sqrt[n]{\frac{f(\mathbf{y})}{\eta}}\right)\right]^n.$$

So (4.15) equivalent to

$$\left(-\min_{\mathbf{y}\in E}\left(|\mathbf{y}-\mathbf{u}| - \sqrt[n]{\frac{f(\mathbf{y})}{\eta}}\right)\right)^n \leq |\mathbf{u}-x|^n. \quad (4.16)$$

Take n-th root to both side of (4.16), which give

$$-\min_{\mathbf{y}\in E}\left(|\mathbf{y}-\mathbf{u}| - \sqrt[n]{\frac{f(\mathbf{y})}{\eta}}\right) \leq |\mathbf{u}-\mathbf{x}|.$$

This equivalent to

$$|\mathbf{u}-\mathbf{x}| + \min_{\mathbf{y}\in E}\left(|\mathbf{y}-\mathbf{u}| - \sqrt[n]{\frac{f(\mathbf{y})}{\eta}}\right) \geq 0.$$

Since $\min(a+b) = a + \min(b)$. Therefore, we need to prove that

$$\min_{\mathbf{y}\in E}\left(|\mathbf{y}-\mathbf{u}| + |\mathbf{u}-\mathbf{x}| - \sqrt[n]{\frac{f(\mathbf{y})}{\eta}}\right) \geq 0. \quad (4.17)$$

The left side of (4.17) equal to

$$\min_{\mathbf{y}\in E}\left(|\mathbf{y}-\mathbf{u}| + |\mathbf{u}-\mathbf{x}| - \sqrt[n]{\frac{f(\mathbf{y})}{\eta}}\right)$$
$$\geq \min_{\mathbf{y}\in E}\left(|\mathbf{y}-\mathbf{x}| - \sqrt[n]{\frac{f(\mathbf{y})}{\eta}}\right)$$
$$= \varpi_{(n,\eta,f)}(\mathbf{x},E) \geq 0.$$

Therefore, (4.14) holds, so that

$$\mathbf{CO}\left[\eta|\cdot-\mathbf{x}|^n - D^n_{(n,\eta,f)}(\cdot,E)\right](\mathbf{x}) = 0 = -C^u_{n,\eta}\left(D^n_{(n,\eta,f)}(\cdot,E)\right)(\mathbf{x}).$$

So that

$$D^n_{(n,\eta,f)}(\mathbf{x},E) \leq C^u_{n,\eta}\left(D^n_{(n,\eta,f)}(\cdot,E)\right)(\mathbf{x}) = 0 \leq C^u_{n,\eta}\left(f\chi_{\mathbf{x}^E}\right)(\mathbf{x}). \quad (4.18)$$

The second case: If the distance-like function $\varpi_{(n,\eta,f)}(\mathbf{x},E) < 0$

$$\varpi_{(n,\eta,f)}(\mathbf{x},E) = \min_{\mathbf{y}\in E}\cdot\left(|\mathbf{y}-\mathbf{x}| - \sqrt[n]{\frac{f(\mathbf{y})}{\eta}}\right)$$
$$= |\mathbf{x}^E - \mathbf{x}| - \sqrt[n]{\frac{f(\mathbf{x}^E)}{\eta}} < 0 \quad (4.19)$$

Since $\mathbf{x}^E$ is the minimum element in the set E then, we have

$$D^n_{(n,\eta,f)}(\mathbf{x},E) = \left[-\sqrt[n]{\eta}\left(|\mathbf{x}^E - \mathbf{x}| - \sqrt[n]{\frac{f(\mathbf{x}^E)}{\eta}}\right)\right]^n.$$

Let the function $f(\mathbf{y})\chi_{\mathbf{x}^E}(\mathbf{y})$ for $\mathbf{y} \in \mathbb{R}^n$ by using Lemma 4.7.

$$C^u_{n,\eta}(f\chi_{\mathbf{x}^E})(\mathbf{y}) = \begin{cases} \eta|\mathbf{y}-\mathbf{x}^E|^n + \dfrac{-nf(\mathbf{x}^E)}{(n-1)}\sqrt[n]{\dfrac{(n-1)\eta}{f(\mathbf{x}^E)}}|\mathbf{y}-\mathbf{x}^E| + f(\mathbf{x}^E) & \text{if } |\mathbf{y}-\mathbf{x}^E| \leq \sqrt[n]{\dfrac{f(\mathbf{x}^E)}{(n-1)\eta}}, \\ 0 & \text{if } |\mathbf{y}-\mathbf{x}^E| \geq \sqrt[n]{\dfrac{f(\mathbf{x}^E)}{(n-1)\eta}}. \end{cases} \quad (4.20)$$

From (4.19), we have $|\mathbf{x}-\mathbf{x}^E| \leq \sqrt[n]{\frac{f(\mathbf{x}^E)}{(n-1)\eta}}$. So that

$$C^u_{n,\eta}(f\chi_{\mathbf{x}^E})(\mathbf{x}) = \eta|\mathbf{x}-\mathbf{x}^E|^n + \frac{-nf(\mathbf{x}^E)}{(n-1)}\sqrt[n]{\frac{(n-1)\eta}{f(\mathbf{x}^E)}}|\mathbf{x}-\mathbf{x}^E| + f(\mathbf{x}^E) \geq D^n_{(n,\eta,f)}(\mathbf{x},E).$$

So that, we have

$$D^n_{(n,\eta,f)}(\mathbf{x},E) \le C^u_{n,\eta}(fX_{\mathbf{x}^E})(\mathbf{x})$$
$$\le C^u_{n,\eta}(fX_E)(\mathbf{x}) \text{ for } \mathbf{x}^E \in E$$

So that, we have $D^n_{(n,\eta,f)}(\mathbf{x},E) \le C^u_{n,\eta}(fX_E)(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m$ and by using the ordering property, we have

$$C^u_{n,\eta}\left(D^n_{(n,\eta,f)}(\cdot,E)\right)(\mathbf{x}) \le C^u_{n,\eta}\left(C^u_{n,\eta}(f\chi_E)\right)(\mathbf{x}) = C^u_{n,\eta}(f\chi_E)(\mathbf{x})$$
(4.21)

The inverse of (4.21) also proved in (4.18) for all $\mathbf{x} \in \mathbb{R}^m$. This completes the proof.

Now, we give result of Hausdorff stability of the n-upper CCT with respect to closed sample sets at a Hölder function f.

Theorem 4.1. Suppose that $f : \mathbb{R}^m \to \mathbb{R}$ is Hölder and positive bounded function. For any two non-empty closed subsets E, K of $\mathbb{R}^m$. Then, we have

$$\left|C^u_{n,\eta}(f\chi_E)(\mathbf{x}) - C^u_{n,\eta}(f\chi_K)(\mathbf{x})\right| \le$$
$$n(\sqrt[n]{M})^{n-1}\sqrt[n]{\eta}\sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}\sqrt{\text{dist}^{\,2}_{\mathcal{H}}(E,K) + \text{dist}^{\frac{2\alpha}{n}}_{\mathcal{H}}(E,K)}$$

Proof. By using Corollary 3.3 we need to prove

$$\left|C^u_{n,\eta}\left(D^n_{(n,\eta,f)}(\mathbf{x},E)\right)(\mathbf{x}) - C^u_{n,\eta}\left(D^n_{(n,\eta,f)}(\mathbf{x},K)\right)(\mathbf{x})\right| \le$$
$$n(\sqrt[n]{M})^{n-1}\sqrt[n]{\eta}\sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}\sqrt{\text{dist}^{\,2}_{\mathcal{H}}(E,K) + \text{dist}^{\frac{2\alpha}{n}}_{\mathcal{H}}(E,K)}$$

By Corollary 4.2 we have,

$$\left|D^n_{(n,\eta,f)}(\mathbf{x},E) - D^n_{(n,\eta),f)}(\mathbf{x},K)\right| \le$$
$$n(\sqrt[n]{M})^{n-1}\sqrt[n]{\eta}\sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}\sqrt{\text{dist}^{\,2}_{\mathcal{H}}(E,K) + \text{dist}^{\frac{2\alpha}{n}}_{\mathcal{H}}(E,K)}$$

Therefore,

$$D^n_{(n,\eta,f)}(\mathbf{x},K) - n(\sqrt[n]{M})^{n-1}\sqrt[n]{\eta}\sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}\sqrt{\text{dist}^{\,2}_{\mathcal{H}}(E,K) + \text{dist}^{\frac{2\alpha}{n}}_{\mathcal{H}}(E,K)} \le$$
$$D^n_{(n,\eta,f)}(\mathbf{x},E) \le$$
$$D^n_{(n,\eta,f)}(\mathbf{x},K) + n(\sqrt[n]{M})^{n-1}\sqrt[n]{\eta}\sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}\sqrt{\text{dist}^{\,2}_{\mathcal{H}}(E,K) + \text{dist}^{\frac{2\alpha}{n}}_{\mathcal{H}}(E,K)}.$$

$$C^u_{n,\eta}(D^n_{(n,\eta,f)}(\cdot,K)(\mathbf{x}) - n(\sqrt[n]{M})^{n-1}\sqrt[n]{\eta}\sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}$$
$$\sqrt{\text{dist}^2_{\mathcal{H}}(E,K) + \text{dist}^{\frac{2\alpha}{n}}_{\mathcal{H}}(E,K))}$$
$$\le C^u_{n,\eta}\left(D^n_{(n,\eta,f)}(\mathbf{x},E)\right) \le$$
$$C^u_{n,\eta}(D^n_{(n,\eta,f)}(\cdot,K)(\mathbf{x}) + n(\sqrt[n]{M})^{n-1}\sqrt[n]{\eta}\sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}$$
$$\sqrt{dist^2_{\mathcal{H}}(E,K) + \text{dist}^{\frac{2\alpha}{n}}_{\mathcal{H}}(E,K))}.$$

By using the ordering and the affine covariance properties of the n-upper

CCT, we have

$$C^u_{n,\eta}(D^n_{(n,\eta,f)}(\cdot,K)(\mathbf{x})$$
$$- n(\sqrt[n]{M})^{n-1}\sqrt[n]{\eta}\sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}\sqrt{\text{dist}^2_{\mathcal{H}}(E,K) + \text{dist}^{\frac{2\alpha}{n}}_{\mathcal{H}}(E,K))}$$
$$\le C^u_{n,\eta}\left(D^n_{(n,\eta,f)}(\mathbf{x},E)\right) \le$$
$$C^u_{n,\eta}(D^n_{(n,\eta,f)}(\cdot,K)(\mathbf{x})$$
$$+ n(\sqrt[n]{M})^{n-1}\sqrt[n]{\eta}\sqrt{1+\left(\frac{L}{\eta}\right)^{\frac{2}{n}}}\sqrt{dist^2_{\mathcal{H}}(E,K) + \text{dist}^{\frac{2\alpha}{n}}_{\mathcal{H}}(E,K))}$$

This complete the proof.

5. Conclusion.

The problem of filling the large gap between bounded Hölder function and its envelope is studied, and the radius of locality of the Hölder function is obtained and it decreases with increasing the convex function parameter η. It is found that the value of the lower CCT depends on the obtained radius of locality and the values of the function in a certain closed ball. Moreover, the Hausdorff stability for the n-upper compensated convex transformations is studied, by considering that the n-upper compensated convex transformation is Hausdorff-Lipschitz continuous with respect to a closed sample sets for the Hölder function f.

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