MANSOURA JOURNAL OF MATHEMATICS
Official Journal of Faculty of Science, Mansoura University, Egypt
E-mail: scimag@mans.edu.eg ISSN:2974-4946

Statistical Analysis of Alpha Power Burr-XII Distribution with Application to Biomedical Data

Nada k. Abd-ElWahab, Dina A. Ramadan, Beih S. El-Desouky

Mathematics Department, Faculty of Science, Mansoura University)

**** Correspondence to:** nadak860@gmail.com, Dinaramadan21@mans.edu.eg, b_desouky@yahoo.com)*

Received:1/5/2023
Accepted: 12/7/2023

Abstract: In this study, we introduce a new set of three parameters α, β, λ for the Burr-XII distribution's alpha power transformation. There are several statistical properties of the APB-XII distribution that are obtained, such as the survival function, hazard rate function, reversed-hazard rate function, cumulative hazard rate function, moments, entropy, quantiles, order statistics, stress-strength analysis, and its explicit expressions. Maximum likelihood method is used to evaluate the unknown parameters. The novel distribution performs better than other distributions, as shown by the goodness-of-fit study of the proposed model. Finally, a set of real-life data is construed and observed that the new model can provide the best fit to bladder cancer patients data than other well-known distributions.

keywords: Burr-XII distribution; Alpha Power Family; Hazard rate function; Estimators of Maximum Likelihood; Moment generating function; Kurtosis and Skewness; Rėnyi-Entropy.

1. Introduction

The Burr type XII distribution introduced by Burr [1] has attracted attention for its remarkable applications in a variety of areas, including reliability, failure time modeling, and acceptance sampling schemes. For example, Wang and Keats [2] employed the maximum likelihood method to estimate the parameters of the Burr-XII distribution using interval estimators. The Burr-XII distribution was used by Abdel-Ghaly et al. [3] in software reliability growth modelling, microelectronics, and reliability. The statistical analysis of the Burr-XII distribution and its relation to other distributions were explored by Zimmer et al. [4]. Moore [5] establishes a confidence range for the form parameter under the Bur-XII distribution's failure censored plan. Maximum likelihood was the method employed by Wu et al. [6] to calculate the point estimator of the parameters of the Burr-XII distribution. AL-Hussaini [7] explored Bayesian predictive density of order statistics based on finite mixture models. AL-Hussaini and Ahmad [8] developed Bayesian interval prediction of future records and according to Kumar [9] derived Marshall Olkin extended Burr-XII distribution's ratio and inverse moments. The Burr-XII distribution has two parameters which is denoted by Burr-XII (β,λ), and it has the following cumulative distribution function (cdf) and probability density function (pdf) of the form

$$G(x;\beta,\lambda) = 1-\left(1+x^{\beta}\right)^{-\lambda}, \qquad x>0,\beta>0,\lambda>0 \quad (1)$$

and

$$g(x;\beta,\lambda) = \lambda\beta x^{\beta-1}(1+x\beta)^{-(\lambda+1)} \quad ,x>0,\beta>0,\lambda>0 \quad (2)$$

where α and β are shape and scale parameters. Recently, Mahdavi and Kundu [10] introduced the new class of distributions known as the alpha power transformation (APT) family. There are extended distributions from the alpha power family, such numerous academics have proposed the APT distribution, for example Nassar et al. [11] extended alpha power Weibull distribution, Ramadan and Magdy [12] studied the alpha power inverse Weibull distribution, Dey et al. [13] generalized exponential distribution was extended in a novel way. and Nadarajah and Okorie [14] extended the generalized exponential, Ihtisham et al. [15] extended the Pareto, Dey et al. [16] extended the inverse Lindley, Zein Eldin et al. [17] extended the inverse Lomax distribution, and so on. The cumulative distribution function (cdf) for APT family($x \in R$) defined as

$$F_{APT}(x) = \begin{cases} \dfrac{\alpha^{F(x)} - 1}{\alpha - 1}, & if\ \alpha > 0\ , \alpha \neq 1 \\ F(x) \quad , & if\ \alpha = 1 \end{cases} \tag{3}$$

$$F_{APT}(x) = \begin{cases} \dfrac{log\ (\alpha)}{\alpha - 1} f(x) \alpha^{F(x)}, & if\ \alpha > 0\ , \alpha \neq 1 \\ F(x) \quad , & if\ \alpha = 1 \end{cases} \tag{4}$$

The remaining sections of the essay are structured as follows. The Alpha Power Burr-XII and its pdf and cdf are introduced In Section 2. APB-XII statistical's properties, such as the moment generating function, quantial function, median, mode, skewness, kurtosis, moments, and so on, were introduced in Section 3. The distribution of the minimum, maximum, and median values of order statistics for APB-XII were provided in Section 4. The renyi-entropy for APBX-II is given in Section 5. We provide APBX-II stress's strength, in Section 6. In Section 7, we employ the asymptotice confidence bounds and maximum likelihood, the most efficient methods of estimation. The APBX-II distribution is applied to actual bladder cancer data in Section 8. We provide some concluding observations in Section 9.

2. Alpha Power Burr-XII Disribution

The random variable X has the alpha power Burr-XII distribution denoted by APB-XII (x, β, λ). Let the cdf and pdf for the Burr-XII distribution with two parameters be $G(x; \beta, \lambda)$ and $g(x; \beta, \lambda)$ for Burr-XII distribution. The Alpha Power Burr-XII (APB-XII) distribution's cdf and pdf are obtained by using G(x) and g(x) from Equations (1) and (2), respectively, in Equations (3) and (4). The cdf of a random variable x that have three parameters APB-XII distribution can be described as

$$F(x; \alpha, \beta, \lambda) = \begin{cases} \dfrac{\alpha^{1-(1+x^{\beta})^{-\lambda}} - 1}{\alpha - 1}, & if\ \alpha > 0\ , \alpha \neq 1 \\ G(X) \quad , & if\ \alpha = 1, \end{cases} \tag{5}$$

where $\alpha, \beta, \lambda > 0$ and $x > 0$.

The corresponding pdf is given as

$$f(x; \alpha, \beta, \lambda) = \begin{cases} \dfrac{log\ (\alpha)}{\alpha - 1} \lambda \beta x^{\beta - 1} (1 + x^{\beta})^{-(\lambda+1)} \alpha^{1-(1+x^{\beta})^{-\lambda}}, & if\ \alpha > 0\ , \alpha \neq 1 \\ g(x) \quad , & if\ \alpha = 1, \end{cases} \tag{6}$$

where $\alpha, \beta, \lambda > 0$ and $x > 0$.

The survival function, $S(x)$ of the APB-XII distribution is attainable as

$$S(x; \alpha, \beta, \lambda) = \frac{(\alpha - 1) - \alpha^{1-(1+x^{\beta})^{-\lambda}} + 1}{\alpha - 1} \quad ; x > 0\ , \alpha, \beta, \lambda > 0. \tag{7}$$

The hazard rate function, $h(x)$, of the APB-XII distribution is expressed as

$$h(x; \alpha, \beta, \lambda) = \frac{log\ (\alpha) \lambda \beta x^{\beta - 1} (1 + x^{\beta})^{-(1+\lambda)} \alpha^{1-(1+x^{\beta})^{-\lambda}}}{(\alpha - 1) - \alpha^{1-(1+x^{\beta})^{-\lambda}} + 1} \quad ; x > 0\ , \alpha, \beta, \lambda > 0. \tag{8}$$

The reversed-hazard rate function, $r(x)$, of the APB-XII distribution is showen as

$$r(x; \alpha, \beta, \lambda) = \frac{\lambda \beta\ log\ (\alpha) x^{\beta - 1} (1 + x^{\beta})^{-(1+\lambda)} \alpha^{1-(1+x^{\beta})^{-\lambda}}}{\alpha^{1-(1+x^{\beta})^{-\lambda}} - 1} \quad ; x > 0\ , \alpha, \beta, \lambda > 0. \tag{9}$$

The APB-XII distribution's cumulative hazard rate function, $H(x)$, is given by

$$H(x; \alpha, \beta, \lambda) = -log \left[\frac{(\alpha - 1) - \alpha^{1-(1+x^{\beta})^{-\lambda}} + 1}{(\alpha - 1)} \right] \quad ; x > 0\ , \alpha, \beta, \lambda > 0 \tag{10}$$

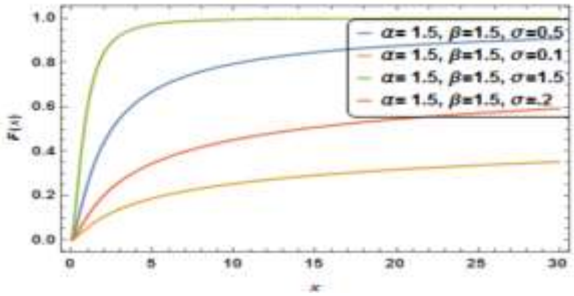

Figure 1: The $F(x)$ of the APBD for for various parameter values.

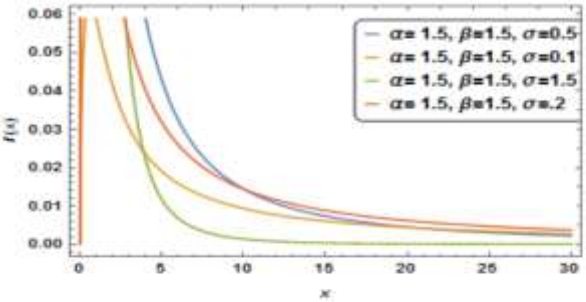

Figure 2: The $f(x)$ of the APBD for various parameter values.

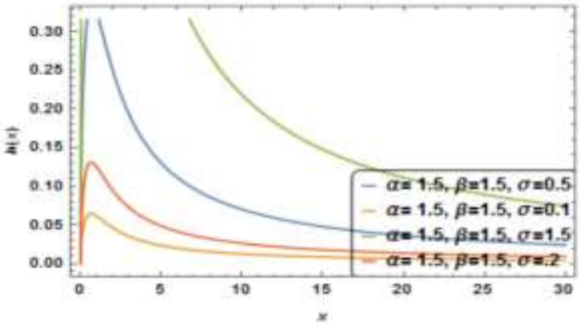

Figure 3: The $h(x)$ of the APBD for various parameter values.

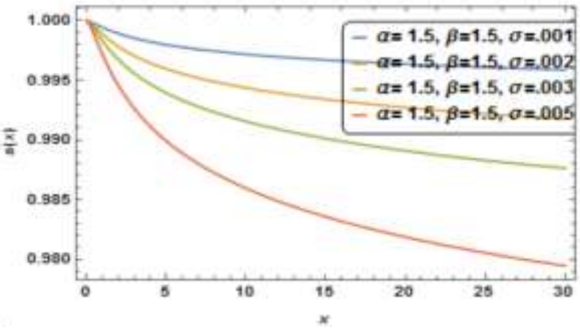

Figure 4: The $S(x)$ of the APBD for various parameter values.

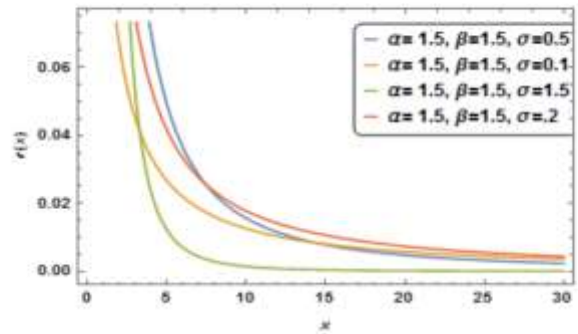

Figure 5: The $r(x)$ of the APBD for various parameter values.

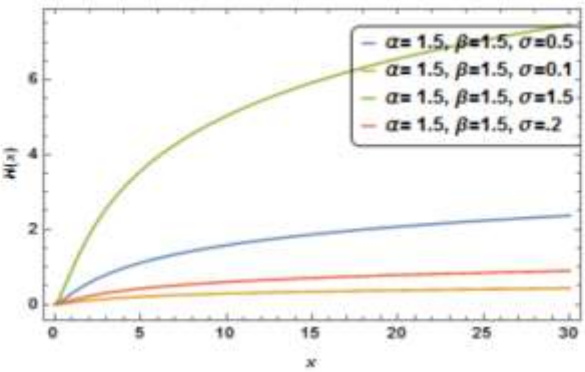

Figure 6: The $H(x)$ of the APBD for various parameter values.

Figures 1-6 display the probability density function (pdf), cumulative distribution function (cdf), survival function, hazard rate function, and inverted hazard rate function of the APB-XIID($x; \alpha, \beta, \lambda$) values for various parameters.

3. Some Statistical Properties

In this section, we provide some of the APB-XII distribution's statistical characteristics.

3.1 Quantile function and Median

There are several important applications for the quantile function, including the ability to produce random variables and determine the skewness, kurtosis, and median. Assume that x is an element of chance drawn from the APB-XII distribution and that the cdf comes from Equation (5), the quantile function of x, is given by

$$x = Q(u) = F^{-1}(u) = F^{-1}\left(\frac{\alpha^{1-(1+x^{\beta})^{-\lambda}}+1}{\alpha-1}\right), \tag{11}$$

where $u\sim$ uniform (0,1) and random numbers can simply be generated from the APB-XII distribution using

$$x = ((1\text{-}(\frac{\ln(u(\alpha-1)+1)}{\ln\alpha})^{\frac{-1}{\lambda}}) - 1))^{\frac{1}{\beta}}. \tag{12}$$

The APB-XII distribution's median may be calculated by setting $u = 0.5$ in Equation (12)

$$median = \left(\left(1-\left(\frac{\ln(0.5(\alpha-1)+1)}{\ln\alpha}\right)^{\frac{-1}{\lambda}}\right)-1\right)^{\frac{1}{\beta}}. \tag{13}$$

3.2 Mode of APB-XII

By differentiating the probability density function of the APB-XII distribution, the distribution's mode is obtained. PDF $f(x)$ in Equation (6) explained a random variable $x > 0$ and equal to zero. Thus, the mode can be provided by

$$\lambda\beta\frac{\alpha}{\alpha-1}((\beta-1)(x^{\beta-2})(1+x^{\beta})^{-(\lambda+1)}(\alpha)^{1-(1+x^{\beta})^{-\lambda}}$$
$$-(\lambda+1)\beta x^{2(\beta+1)}(1+x^{\beta})^{-(\lambda+2)}(\alpha)^{1-(1+x^{\beta})^{-\lambda}}+\lambda\beta x^{2(\beta+1)}(1+x^{\beta})^{-2(\lambda+1)}$$
$$\log(\alpha) = 0. \tag{14}$$

3.3 Skewness and Kurtosis

Because the moment does not always occur, it serves as a common measure for a distribution’s skewness and kurtosis. Therefore, we employed the skewness and kurtosis because it is widely known how unreliable a common way to assess kurtosis is. Kenney and Keeping [18] described the skewness in

$$S_k = \frac{q_{(0.75)} - 2q_{(0.5)} + q_{(0.25)}}{q_{(0.75)} - q_{(0.25)}}. \quad (15)$$

The Moors quantile based Kurtosis as

$$K_k = \frac{q_{(0.875)} - q_{(0.625)} - q_{(0.375)} - q_{(0.125)}}{q_{(0.75)} - q_{(0.25)}}. \quad (16)$$

A notice of S_k is denoted by the direction of the distribution's skewness, $S_k < 0$ denoting to right skewness, $S_k = 0$ denote to symmetric and $S_k > 0$ denote to left skewness. The larger the value, the heavier the distribution's tail is.

3.4 Moments

We will discuss rth moment for APB-XII distribution, for any statistical analysis applications moments are very important. The rth moment of the Alpha Power Burr-XII (APB-XII) distribute is derived as

$$\mu_r = \int_0^\infty x^r f(x; \alpha, \beta, \lambda) dx. \quad (17)$$

By substituting from Equation (6) into Equation (17), we get

$$\mu_r = \int_0^\infty x^r \frac{\log(\alpha)}{\alpha - 1} \lambda\beta x^{\beta-1} (1 + x^\beta)^{-(\lambda+1)} \alpha^{1-(1+x^\beta)^{-\lambda}} dx. \quad (18)$$

By using Alpha Power Exponential formula,

$$\alpha^{1-(1+x^\beta)^{-\lambda}} = \sum_{k=0}^{\infty} \frac{(\ln(\alpha))^k}{k!} \left(1 - (1 + x^\beta)^{-\lambda}\right)^K$$

, then

$$\mu_r = \sum_{k=0}^{\infty} \frac{\lambda\beta(\log(\alpha))^{k+1}}{k!(\alpha - 1)} \int_0^\infty x^{r+\beta-1} (1 + x^\beta)^{-(\lambda+1)} \left(1 - (1 + x^\beta)^{-\lambda}\right)^k dx. \quad (19)$$

By using Binomial expansion negative power,

$$1 - (1 + x^\beta)^{-\lambda})^k = \Sigma \binom{k}{i} (-1)^i (1 + x^\beta)^{-i\lambda}$$

,then

$$\mu_r = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i \lambda\beta(\log(\alpha))^{k+1}}{k!(\alpha - 1)} \int_0^\infty x^{r+\beta-1} (1 + x^\beta)^{-(i+1)\lambda-1} dx. \quad (20)$$

Let $u = x^\beta$ and $x = u^{\frac{1}{\beta}}$ then $\frac{1}{\beta} = u^{\frac{1}{\beta}-1} du$, then

$$\mu_r = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i \lambda(\log(\alpha))^{k+1}}{k!(\alpha - 1)} \int_0^\infty u^{\frac{r+\beta-1}{\beta}} (1 + u)^{-(i+1)\lambda-1} u^{\frac{1}{\beta}-1} du$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i \lambda(\log(\alpha))^{k+1}}{k!(\alpha - 1)} \beta\left(\frac{-r}{\beta} + 1, (i + 1)\lambda + \frac{r}{\beta}\right). \quad (21)$$

where $\beta(.,.)$ is incomplete Beta function.

3.5 Moment generating function

The APB-XII distribution's moment generating function (MGF) may be expressed as

$$M_x(t) = \int_0^\infty e^{xt} f(x; \alpha, \beta, \lambda) dx. \quad (22)$$

By using the exponential function, $e^{xt} \sum_{r=1}^{\infty} \frac{x^r t^r}{r!}$,

$$M_x(t) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{\lambda\beta(\log(\alpha))^{k+1}}{k!(\alpha - 1)} \beta\left(\frac{-r}{\beta} + 1, (i + 1)\lambda + \frac{r}{\beta}\right) \quad (23)$$

4. Order Statistics

Assuming that $x_{(1)}, x_{(2)}, x_{(3)}, \ldots. \; x_{(n)}$ are order statistics of a random sample follows continuous distribution with cdf $F(x)$ and pdf $f(x)$, then the pdf of x_k is given by

$$f_{(k:n)}(x) = \frac{n!}{(k - 1)!(n - k)!} f(x)[F(x)]^{k-1} [1 - F(x)]^{n-k}, k = 1,2,3, \ldots, n. \quad (24)$$

If x is a random variable in the APB-XII distribution, the density function of the order statistics would be

$$f_{(k:n)}(x) = \frac{n!}{(k-1)!\,(n-k)!}\frac{\log[\alpha]}{\alpha-1}\lambda\beta x^{\beta-1} \left(1+x^{\beta}\right)^{-(\lambda+1)}\alpha^{1-(1+x^{\beta})^{-\lambda}} \left[\frac{\alpha^{1-(1+x^{\beta})^{-\lambda}}-1}{\alpha-1}\right]^{k-1}\left[1-\frac{\alpha^{1-(1+x^{\beta})^{-\lambda}}-1}{\alpha-1}\right]^{n-k}. \quad (25)$$

If $k=1$, the pdf of the order statistics is

$$f_{(1:n)}(x) = n\left[\frac{\log[\alpha]}{\alpha-1}\lambda\beta x^{\beta-1}\left(1+x^{\beta}\right)^{-(\lambda+1)}\alpha^{1-(1+x^{\beta})^{-\lambda}}\right] \times\left[1-\frac{\alpha^{1-(1+x^{\beta})^{-\lambda}}-1}{\alpha-1}\right]^{n-1}. \quad (26)$$

If $k=n$, the pdf of the order statistics is

$$f_{(n:n)}(x) = n\left[\frac{\log[\alpha]}{\alpha-1}\lambda\beta x^{\beta-1}\left(1+x^{\beta}\right)^{-(\lambda+1)}\alpha^{1-(1+x^{\beta})^{-\lambda}}\right] \times\left[\frac{\alpha^{1-(1+x^{\beta})^{-\lambda}}-1}{\alpha-1}\right]^{n-1}. \quad (27)$$

4.1 Distribution of minimum, maximum and median

Let $x_{(1)}, x_{(2)}, x_{(3)}, \ldots.\ x_{(n)}$ be independent; identically distributed random variables from

APB-XII distribution, then

$$f_{max}(x) = \frac{d}{dx}\left(F(x)\right)^{n} = n\left(F(x)\right)^{n-1}f(x). \quad (28)$$

$$f_{max}(x) = n\left[\frac{\alpha^{1-(1+x^{\beta})^{-\lambda}}-1}{\alpha-1}\right]^{n-1}\frac{\log(\alpha)}{\alpha-1}\lambda\beta x^{\beta-1} \left(1+x^{\beta}\right)^{-(\lambda+1)}\alpha^{1-(1+x^{\beta})^{-\lambda}}. \quad (29)$$

$$f_{min}(x) = -\frac{d}{dx}\left(1-F(x)\right)^{n} = n\left(1-F(x)\right)^{n-1}f(x). \quad (30)$$

$$f_{min}(x) = n\left[1-\frac{\alpha^{1-(1+x^{\beta})^{-\lambda}}-1}{\alpha-1}\right]^{n-1} \frac{\log(\alpha)}{\alpha-1}\lambda\beta x^{\beta-1}\left(1+x^{\beta}\right)^{-(\lambda+1)}\alpha^{1-(1+x^{\beta})^{-\lambda}}. \quad (31)$$

$$f_{med}(x) = -\frac{d}{dx}\left(1-F(x)\right)^{n} = \frac{(2m+1)!}{m!\,m!}\left(F(x)\right)^{m}(1\text{-}F(x))^{m}f(x) \quad (32)$$

$$f_{med}(x) = \frac{(2m+1)!}{m!\,m!}\left(\frac{\alpha^{1-(1+x^{\beta})^{-\lambda}}-1}{\alpha-1}\right)^{m} \left(1-\frac{\alpha^{1-(1+x^{\beta})^{-\lambda}}-1}{\alpha-1}\right)^{m} \left[\frac{\log[\alpha]}{\alpha-1}\lambda\beta x^{\beta-1}(1+x^{\beta})^{-(\lambda+1)}\alpha^{1-(1+x^{\beta})^{-\lambda}}\right]. \quad (33)$$

5 Rẻnyi-Entropy

The Rẻnyi-Entropy is a quantity that generalizes several concepts of entropy. It is especially significant in ecology, statistics, and quantum information, it is defined as

$$Re(x) = \frac{1}{1-R}\log\left(\int_0^{\infty} f(x)^{R}dx\right). \quad (34)$$

By substituting from Equation (6) into Equation (27), we get

$$Re(x) = \frac{1}{1-R}\log\left(\int_0^{\infty}\left(\frac{\log[\alpha]}{\alpha-1}\lambda\beta x^{\beta-1}\left(1+x^{\beta}\right)^{-(\lambda+1)}\alpha^{1-(1+x^{\beta})^{-\lambda}}\right)^{R}dx\right)$$

$$= \frac{1}{1-R}\times\log\left[\frac{\log(\alpha)}{\alpha-1}\lambda\beta\right]^{R}\times \left[\int_0^{\infty}\left(x^{R(\beta-1)}\times\left(1+x^{\beta}\right)^{-R(\lambda+1)}\times\alpha^{R\left(1-(1+x^{\beta})^{-\lambda}\right)}\right)dx\right], \quad (35)$$

where,

$$\alpha^{R[1-(1+x^{\beta})^{-\lambda}]} = \sum_{k=0}^{\infty} \frac{(\log(\alpha)R)^k}{k!} (1-1+x^{\beta})^{-\lambda})^k.$$

Then

$Re\,(x) =$

$$\frac{1}{1-R} \times \log\left(\frac{\log(\alpha)}{\alpha-1}\lambda\beta\right)^R$$

$$\times \log \sum_{k=0}^{\infty} \frac{(\log(\alpha)R)^k}{k!}$$

$$\times \log\left(\int_0^{\infty}\left(x^{R(\beta-1)} \times (1+x^{\beta})^{-R(\lambda+1)} \times\right.\right.$$

$$\left.\left.(1-(1+x^{\beta})^{-\lambda})^k\right)dx\right) \quad (36)$$

where

$$\left(1-(1+x^{\beta})^{-\lambda}\right)^k = \sum_{k=0}^{\infty} \binom{k}{i} (-1)^i (1+x^{\beta})^{-i\lambda}$$

Then

$$Re\,(x) = \frac{1}{1-R} \times \log\left(\frac{\log[\alpha]}{\alpha-1}\lambda\beta\right)^R$$

$$\times \log\left(\sum_{k=0}^{\infty}\sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{(\log(\alpha)R)^k}{k!}\right) \quad (37)$$

The final of $Re\,(x)$ is given by

$$Re\,(x) = \frac{1}{1-R} \times \log\left(\frac{\log[\alpha]}{\alpha-1}\lambda\beta\right)^R$$

$$\times \log\left(\sum_{k=0}^{\infty}\sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{(\log(\alpha)R)^k}{k!}\right) \quad (38)$$

6. Stress-Strength

Stress-Strength System lifetimes are described by a measure called stress-strength reliability. Strength variable X and stress variable Y are both present in this reliability model. Numerous statisticians have studied estimation studies of dependability systems using both complete and censored samples from various models, such as the APB-XII distribution.

$$R(x) = \int_0^{\infty} F(x,\alpha_1,\beta,\lambda_1) f(x,\alpha_2,\beta,\lambda_2)dx$$

$$= \int_0^{\infty} \frac{\alpha_1^{(1-(1+x^{\beta}))^{\lambda_1}} - 1}{\alpha_1 - 1}$$

$$\left(\frac{\log[\alpha_2]}{\alpha_2-1}\lambda_2\beta x^{\beta-1}(1+x^{\beta})^{-(\lambda_2+1)} \alpha_2^{1-(1+x^{\beta})^{-\lambda_2}}\right)dx$$

$$= \frac{\lambda_2\beta\log(\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}$$

$$\int_0^{\infty}\left(\alpha_1^{(1-(1+x^{\beta}))^{-\lambda_1}}\right)x^{\beta-1}$$

$$(1+x^{\beta})^{-(\lambda_2+1)} \alpha_2^{1-(1+x^{\beta})^{-\lambda_2}} dx - \frac{1}{\alpha_1-1}. \quad (39)$$

Where

$$(1\text{-}(1+x^{\beta}))^{-\lambda} = \sum_{i=0}^{\infty} \binom{-\lambda}{i} (-1)^i (1+x^{\beta})^i$$

and

$$\alpha^{1-(1+x^{\beta})^{-\lambda}} = \sum_{k=0}^{\infty} \frac{(\ln(\alpha))^k}{k!}$$

$$\left(1-(1+x^{\beta})^{-\lambda}\right)^k.$$

Then

$$\mathrm{R}(x) = \frac{\lambda_2\beta\log(\alpha_2)}{(\alpha_1-1)(\alpha_2-1)}$$

$$\sum_{k=0}^{\infty} \frac{(\ln(\alpha_2))^k}{k!}\frac{(\ln(\alpha_2))^k}{k!}$$

$$\int_0^{\infty}\left(1-(1+x^{\beta})^{-\lambda_1}\right)^k x^{\beta-1}(1+x^{\beta})^{-(\lambda_2+1)}$$

$$\left(1-(1+x^{\beta})^{-\lambda_2}\right)^k dx$$

The final of $R\,(x)$ is given by

$$R(x) = \frac{\lambda_2\log(\alpha_2)}{(\alpha_1-1)(\alpha_2-1)} \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{2i} \binom{k}{i}^2 \frac{(\ln(\alpha_1))^k}{k!}\frac{(\ln(\alpha_2))^k}{k!}$$

$$\left[\left(\frac{(\alpha_2-1)-(i(\lambda_1+\lambda_2)+\lambda_2}{(\alpha_2-1)(i(\lambda_1+\lambda_2)+\lambda_2}\right)\right]. \quad (40)$$

7. Parameter Estimation

The maximum likelihood approach is the most popular for estimating unknown parameters in probability distributions.

Additionally, confidence intervals can be calculated using the MLE s. Using the method of maximum likelihood based on a whole sample, point and interval estimation of the unknown parameters of the Alpha Power Burr-XII distribution are constructed in this section.

Let $x_{(1)}, x_{(2)}, x_{(3)}, \ldots.\ x_{(n)}$ denote to the If you take a random sample of all the data from the APB-XII distribution, you can find the likelihood function by

$$L = \prod_{i=0}^{n} f\,(x_i;\alpha,\beta,\lambda). \qquad (41)$$

substituting from Equation (6) into Equation (41), we have

$$L = \prod_{i=0}^{n} \frac{\log(\alpha)}{\alpha-1}\lambda\beta\, x^{\beta-1}\left(1+x^{\beta}\right)^{-(\lambda+1)} \alpha^{1-\left(1+x^{\beta}\right)^{\lambda}}.$$

The parameters α, β and λ associated log-likelihood function is

$$l = n\log\big(log\,(\alpha)\big) + n\log(\lambda) + n\log(\beta) - n\log(\alpha) + (\beta-1)\sum_{i=0}^{m}\log(x_i) - (\lambda+1)\sum_{i=0}^{m}\log\left(1+x_i^{\beta}\right) + \log(\alpha)\left(n - \sum_{i=0}^{\infty}\left(1+x_i^{\beta}\right)^{-\lambda}\right). \qquad (42)$$

So, by differentiating and maximizing the log-likelihood function in Equation (42), with respect to unknow parameters and are obtained as follows:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha\log\,(\alpha)} - \frac{n}{\alpha} - \sum_{i=0}^{\infty}\log\left(1+x_i^{\beta}\right) + \alpha^{-1}\left(\frac{n-\sum_{i=0}^{\infty}\left(1+x_i^{\beta}\right)^{-\lambda}}{\alpha}\right). \qquad (43)$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=0}^{\infty}\log\,(x_i) - (\lambda+1)\sum_{i=0}^{\infty}\frac{\left(x_i^{\beta}\right)\log\,(x_i)}{1+x_i^{\beta}} + \lambda\log(x)\log(\alpha)\sum_{i=0}^{\infty} x_i^{\beta}\left(1+x_i^{\beta}\right)^{-\lambda-1}, \qquad (44)$$

and $\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=0}^{\infty}\log\left(1+x_i^{\beta}\right)$

$$+\log\,(\alpha)\sum_{i=0}^{\infty}\left(1+x_i^{\beta}\right)^{-\lambda}\log\left(1+x_i^{\beta}\right). \qquad (45)$$

The MLEs can be obtained by solving the nonlinear Equations (43) – (45); numerically for α, β and λ.

7.1 Asymptotic confidence bounds

Using the variance covariance matrix, we are able to determine the MLES's asymptotic variance and covariance for the three parameters. (I^{-1}) and it is defined as follows

$$I^{-1} = \begin{pmatrix} -\frac{\partial^2 \iota}{\partial \alpha^2} & -\frac{\partial^2 \iota}{\partial \lambda\partial \beta} & -\frac{\partial^2 \iota}{\partial \alpha\partial \lambda} \\ -\frac{\partial^2 \iota}{\partial \beta\partial \alpha} & -\frac{\partial^2 \iota}{\partial \beta^2} & -\frac{\partial^2 \iota}{\partial \beta\partial \lambda} \\ -\frac{\partial^2 \iota}{\partial \lambda\partial \alpha} & -\frac{\partial^2 \iota}{\partial \lambda\partial \beta} & -\frac{\partial^2 \iota}{\partial \lambda^2} \end{pmatrix}^{-1} = \begin{pmatrix} var(\alpha) & cov(\alpha,\beta) & cov(\alpha,\lambda) \\ cov(\beta,\alpha) & var(\beta) & cov(\beta,\lambda) \\ cov(\lambda,\alpha) & cov(\lambda,\beta) & var(\lambda) \end{pmatrix}. \qquad (46)$$

Where

$$\frac{\partial^2 l}{\partial \alpha^2} = \frac{-n}{\alpha^2\log\,(\alpha)^2} - \frac{n}{\alpha^2\log\,(\alpha)} + \frac{n}{\alpha^2} - \alpha^2\left(n - \sum_{i=0}^{\infty}\left(1+x_i^{\beta}\right)^{-\lambda}\right). \qquad (47)$$

$$\frac{\partial^2 l}{\partial \alpha\partial \beta} = -\sum_{i=0}^{\infty}\frac{x_i^{\beta}\log(x_i)}{1+x_i^{\beta}} + \alpha^{-1}\left(\lambda\,log\,(\alpha)\sum_{i=0}^{\infty}x_i^{\beta}\left(1+x_i^{\beta}\right)^{-\lambda-1}\right). \qquad (48)$$

$$\frac{\partial^2 l}{\partial \alpha\partial \lambda} = \alpha^{-1}\left(\sum_{i=0}^{\infty}\left(1+x_i^{\beta}\right)^{-\lambda} log\left(1+x_i^{\beta}\right)\right). \qquad (49)$$

$$\frac{\partial^2 l}{\partial \beta^2} = \frac{-n}{\beta^2} - \sum_{i=0}^{\infty} \frac{x_i^{2\beta}(1+\lambda)\log(x_i)^2}{\left(1+x_i^{\beta}\right)^2}$$
$$+\sum_{i=0}^{\infty} \frac{x_i^{\beta}(1+\lambda)\log{(x_i)}^2}{\left(1+x_i^{\beta}\right)}$$
$$+\sum_{i=0}^{\infty} x_i^{\beta}\left(1+x_i^{\beta}\right)^{-1-\lambda}\lambda\log(x_i)^2\log(\alpha)$$
$$+\sum_{i=0}^{\infty} x_i^{2\beta}\left(1+x_i^{\beta}\right)^{-2-\lambda}$$
$$(-1-\lambda)\lambda\log{(x_i)}^2\log{(\alpha)}. \tag{50}$$

$$\frac{\partial^2 l}{\partial \beta \partial \lambda} = \sum_{i=0}^{\infty} x_i^{\beta}\left(1+x_i^{\beta}\right)^{-1-\lambda}\log{(x_i)}\log{(\alpha)} -$$
$$\sum_{i=0}^{\infty} x_i^{\beta}\left(1+x_i^{\beta}\right)^{-1-\lambda}\lambda\log(x_i)\log(\alpha)$$
$$\log{\left(1+x_i^{\beta}\right)} + \sum_{i=0}^{\infty} \frac{\left(x_i^{\beta}\right)\log{(x_i)}}{1+x_i^{\beta}}, \tag{51}$$

and

$$\frac{\partial^2 l}{\partial \lambda^2} = \frac{-n}{\lambda^2} - \sum_{i=0}^{\infty}\left(1+x^{\beta}\right)^{-\lambda}$$
$$\log{\left(1+x^{\beta}\right)}^2\log{(\alpha)}. \tag{52}$$

The asymptotic variance-covariance matrix V of the estimate $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ is obtained by inverting the Hessian matrix. Approximate $100(1-\delta)\%$ two-sided confidence intervals for α, β and λ are given by

$$\hat{\alpha} \pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{\alpha})}, \hat{\beta} \pm z_{\frac{\delta}{2}}\sqrt{var(\hat{\beta})}$$
$$, \hat{\lambda} \pm z_{\frac{\delta}{2}}\sqrt{var(\hat{\lambda})}.$$

where $Z_{\frac{\delta}{2}}$ is percentile of the standard normal distribution with the right-tail probability $\frac{\delta}{2}$

8. Applications

All of the following real data set, which shows the lengths of remission in more than month for a sample has (128) bladder cancer patients, in this section we will be used to highlight the usefulness of the APB-XII distribution. Sometimes the tumor invades the bladder muscle, see Lee and Wang [19]. The data is following

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98	6.97
2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26	9.47
2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81	2.62	3.82	5.32	7.32	10.06
3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66	15.96
4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75	4.26	5.41	7.63	17.12	46.12
5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64	17.36	1.40	3.02
11.79	18.10	1.46	4.40	5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37
4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76	12.07	21.73	2.07	3.36	6.93
13.29	0.40	25.82	0.51	32.15	2.64	1.05	2.69	2.83	4.33	5.71	7.93	2.02
12.63	22.69	9.02	14.24	14.77	36.66	1.26	4.34	12.02	8.65	3.31		

This data was used by Kumar et al. [9], Chandra [20], El-Gohary et al. [21], De Andrade and Zea [22], Selim [23] and Buzaridah [24]. We fitted the above-mentioned using MLEs to the alpha power Burr-XII (APB-XII) distribution. The MLEs and the standard errors for IE, IW, INH, IGPW, FRL-IL and APB-XII distributions are shows in Table 1. Kolmogorov-Smirnov (K-S), -Log-likelihood (-L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), and Bayesian Information Criterion (BIC) were used to compare the fitted models. The model that receives the lowest values for the information criterion and goodness-of-fit statistics is the best model according to this criterion. The APB-XII model is therefore superior to the other models in terms of fit and performance, as shown by the numerical findings in Table 2. The empirical and fitted cumulative for the APB-XII distribution are shown in Figure 7. Additionally, these graphics show that, when we makes a compare to the other models, the APB-XII distribution was the best fit to our data. As a result, the Flexible Reduced Logarithmic - Inverse Lomax distribution and other well-known models can be employed as potential alternatives to the APB-XII distribution.

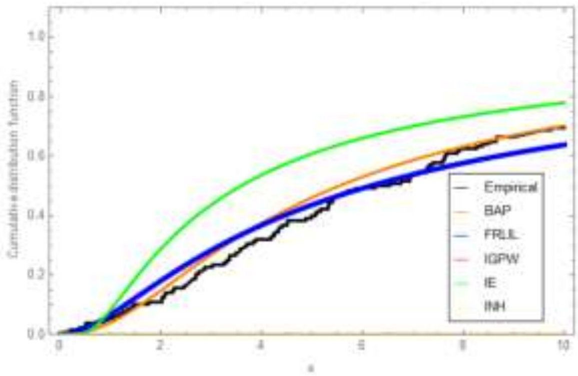

Figure 7: Fitted cumulative for the APBD

$Table1: The\ estimates\ of\ \alpha, \beta\ and\ \lambda$			
$Model$	α	β	λ
IL	$-$	0.6248	$-$
IR	$-$	0.0395	$-$
IW	$-$	0.8625	0.7585
INH	0.5130	2.6070	$-$
$IGPW$	0.2356	12.3717	2.0876
$FRL-IL$	0.006725	1.95653	2.25472
APB	1129.83	1.11214	1.13519

$Table\ 2: The\ estimates\ of\ the\ goodness-of-fit\ for\ data$					
$Model$	$K-S$	$-L$	AIC	$CAIC$	BIC
IL	0.2311	460.382	922.765	922.796	925.617
IR	0.7502	774.342	1550.683	1550.715	1553.535
IW	0.1408	444.001	892.002	892.098	897.706
INH	0.1636	431.059	866.118	866.214	871.822
$IGPW$	0.1364	426.91	859.819	860.013	868.375
$FRL-IL$	0.111778	425.34	856.681	856.874	865.237
APB	0.0713398	417.394	840.789	840.982	849.345

9. Conclusion

In this paper a new three parameters (α, β, λ), for the Alpha Power Burr-XII transformation have introduced. The median, the hazard rate function, the surviver function, reversed hazard rate function, moments, quantile function, median, and the order statistical are some of the derivations of statistical properties. The parameters (α, β, λ) of the model are estimated by using the maximum likelihood method. The application has proven and make as sure that the suggested distribution is advantageous and superior for handling dependability data. Additionally, the picture graphically demonstrates how well APB-XII fits our data. In future work will application a progressive type-II censored data.

10. References

1. Burr, I. W. (1942). Cumulative frequency functions, The annals of mathematical statistics, **13**, 215-232.
2. Wang, F. K. and Keats, J. B. (1996). Maximum likelihood estimation of the Burr type XII parameters with censored and uncensored data, Microelectronics and Reliability,**36**, 359-362.
3. Abdel-Ghaly, A. A., Al-Dayian, G. R. and Al-Kashkari, F. H. (1997). The use of Burr type XII distribution on software reliability growth modeling, Microelectronics and reliability, 37, 305-313.
4. Zimmer, W. J., Keats, J. B. and Wang, F. K. (1998). The Burr XII distribution in reliability analysis, *Journal of Quality Technology*, **30**, 386-394.
5. Moors, J. J. A. (1998). A quantile alternative for kurtosis, The Statistician, **37**, 1, 25-32.
6. Wu, S. J., Chen, Y. J. and Chang, C. T. (2007). Statistical inference based on progressively censored samples with random removals from the Burr type XII distribution, *Journal of Statistical Computation and Simulations*, 77, 19-27.
7. AL-Hussaini, E. K. (2003). Bayesian predictive density of order statistics based on finite mixture models, *Journal of statistical planning and inference*, 113, 15-24.
8. AL-Hussaini, E. K. and Ahmad, A. A. (2003). On bayesian interval prediction of future records, Test, 12, 79-99.
9. Kumar, D., Singh,U. and Singh,S., K. (2015). A method of proposing new distribution and its application to bladder cancer patients data, *Journal of statistics applications and probability letters*, **2**, 3, 235-245.
10. Mahdavi, A. and Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution Commun, Statistics, Theory Methods, 46 ,13, 6543-6557.
11. Nassar, M., Alzaatreh, A., Mead, M.,and Abo-Kasem, O., (2017). Alpha power Weibull distribution: properties and applications, Commun Stat Theory Methods, 46,20,10236-10252.

12. Ramadan, D. A. and Magdy, W. A. (2018). On the alpha power inverse Weibull. *Int. J. Comput. Appl* **181**, (0975-8887).
13. Dey, S.,Alzaatreh, A. ,Zhang, C. and Kumar, D. (2017) . A new extension of generalized exponential distribution, The scientific *journal Ozone Science and Engineering*, 39,4, 273-285.
14. Nadarajah, S. and Okorie, I.E. (2018) . On the moments of the alpha power transformed generalized exponential distribution, The scientific *journal ozone science and engineering*, **40,** 330-335.
15. Ihtisham, S., Khalil, A., Manzoor, S,S.A. and Ali, A. (2019). Alpha-power Pareto distribution: its properties and applications, PLoS ONE, **14**,6, 1-15.
16. Dey,S., Nassar,M. and Kumar, D. (2019). Alpha power transformed inverse Lindley distribution: a distribution with an upside-down bathtub-shaped hazard function, *Journal of computational and applied mathematics*, **348**, 130-145.
17. ZeinEldin, R.A., Ahsan ul Haq,M. , Hashmi,S. and Elsehety,M.(2020). Alpha power transformed inverse lomax distribution with different methods of estimation and applications, Complexity, 2020, 15.
18. Kenney, J. and Keeping, E. (1962). Mathematics of statistics, Nostrand company, Princeton, 1,6
19. Lee, E. T. and Wang, J. (2003). Statistical methods for survival data analysis, John Wiley and Sons, 476,534.
20. Chandra, N. and Sen, S. (2017). The quasi xgamma distribution with application in bladder cancerdata, *Journal of Data Science,* **15**, 61-76.
21. El-Gohary, A., El-Bassiouny, A. and El-Morshedy, M. (2015). Inverse aexible weibull extension distribution, International *Journal of Computer Applications*, 115, 2,46-51.
22. De Andrade, T. A. and Zea, L. M. (2018). The exponentiated generalized extended Pareto distribution, *Journal of Data Science*, **16**, 4, 781-800.
23. Lawless, J. F. (2011). Statistical Models and Methods for Lifetime Data, John Wiley and Sons, New York. 362,664.
24. Selim, M. A. (2019). Some theoretical and computational aspects of the inverse generalized power Weibull distribution, *Journal of Data Science*, **17,** 4, 742 - 755.
25. Buzaridah,M. M. , Ramadan,D. A. andEl-Desouky, B. S. (2021). Flexible reduced logarithmic-inverse lomax distribution with application for bladder cancer, *Open Journal of Modelling and Simulation***,** **9**, 4, 351ñ369.