HEAT TRANSFER TO THE TURBULENT BOUNDARY LAYER OVER A FLAT PLATE WITH NON-UNIFORM SURFACE TEMPRATURE

BY

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ABSTRACT:

Theoretical analysis is presented for heat transfer to an essentially constant turbulent boundary layer from the surface of a flat plate having non-uniform temperature. The solution is based on the hydrodynamic theory of heat transfer and an expression for the local Nusselt number is derived. It is shown by calculation that with surface temperature increasing in the direction of the flow, the rate of heat transfer to the boundary layer is greater than that what would be predicted assuming quasi-uniform surface temperature and vice-versa. Also, the present analysis is shown to agree well with some existing experimental data.

NOMENCLATURE:

a , molecular thermal diffusivity = $K/\rho C_p$;

a_T , turbulent thermal diffusivity = $K_T/\rho C_p$;

C_p , specific heat at constant pressure;

h , local heat transfer coefficient;

K , fluid molecular thermal conductivity;

K_T , fluid turbulent thermal conductivity;

Nu_x, local Nusselt number = $h\,x/K$;

Pr , prandtl number = $\mu\, c_p/K$;

q , density of heat flux from plate;

q_c , density of heat flux from plate assuming quasi-uniform surface temperature;

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q_s , density of heat flux across the laminar sub-layer;
q_T , density of heat flux across the turbulent part of the boundary layer;
Re_x , local Reynolds number $= \rho u_\infty x / \mu$;
S_s , shearing stress in the laminar sub-layer;
S_T , shearing stress in the turbulent part of the turbulent boundary layer;
St_x , local Stanton number $= h/\rho u_\infty C_p = Nu_x / Re_x . Pr$;
t , temperature of the fluid in thermal boundary layer;
$\bar{t}$, average temperature at any point in the turbulent part of the turbulent boundary layer;
t_s , fluid temperature at the outer edge of the laminar sub-layer;
t_w , temperature of the plate surface;
t_∞ , fluid temperature at infinity;
u , velocity component along axis x;
$\bar{u}$, average velocity component at any point in the turbulent part of the turbulent boundary layer;
u_s , velocity component at the outer edge of the laminar sub-layer;
u_∞ , free-stream velocity;
x , linear coordination calculated along the plate from the front edge;
y , linear coordination calculated normal to the plate;
δ_h , thickness of hydradynamic boundary layer;
$\delta_{h,s}$, thickness of hydrodynamic laminar sub-layer;
δ_t , thickness of thermal boundary layer;
$\delta_{t,s}$, thickness of themal laminar sub-layer;
$\delta_{t,s,c}$, thickness of themal laminar sub-layer with uniform surface temperature;
$\lambda_{t,x}$, ratio of heat fluxes q/q_c; and determined by Eq. (21);
$\epsilon_{t,x}$, coefficient of surface temperature non-uniformity, and determined by equation (14);
μ , dynamic viscosity $= \rho \nu$;
μ_T , coefficient of turbulent transmission of momentum $= \rho \nu_T$;
ν , kinematic viscosity;
ν_T , turbulent kinematic viscosity;
ρ , fluid density.

INTRODUCTION:

The case of the heat transfer to a fluid flowing over a plate with an arbitrary variation of surface temperature t(x) is widely encountered in engineering applications, and its calculation presents many difficults. In many cases, these are created by the fact that the local heat flux is by no means determined solely by the local temperature difference $[t_w(x)-t_\infty]$.

The case of the laminar boundary layer in the presence of a surface temperature which is arbitrary distributed along the wall was worked out in [1] , [2] , [3] and others. Moretti and Kays[4] have studied experimentally heat transfer to turbulent boundary layer with varying surface temperature and varying freestreem velocity, and for the purpose of the comparison with an analytical solution they have worked out an empirical modification of the Ambrok solution[2] which has studied the effect of the surface temperature variability on heat exchange in laminar boundary layer. They have obtained a relation for heat transfer to the turbulent boundary layer over a flat plate with a step-wise variation of surface temperature as follows:

$$St(\xi, x) = 0.0295\, Pr^{-0.4}\, Re^{-0.2} \left[1 - (\xi / x)\right]^{-0.12}, \qquad (a)$$

where ξ is the distance from beging of plate to a step in surface temperature.

The prosent analysis is concerned with the ateady heat transfer to turbulent boundary layer over a flat plate whose surface temperature is arbitrarily non-uniform. A solution to heat transfer equation is proposed and a practical example is given in support of the validity of the present solution.

ANALYSIS:

The theoretical study of the heat transfer to the turbulent boundary layer is based on the hydrodynamic theory of heat transfer. This makes it possible to drive calculating formulae for heat transfer based on friction calculation. The hydrodynamic theory describes the mechanism of the turbulent heat transfer sufficiently well for practical purposes.

The system shown schematically in Fig. (1) consists of a semi-infinite flat plate whose surface temperature t_w is assumed to vary with x arbitcary. The free stream velocity u_∞ and temperature t_∞ are taken to be constant and the flow is parallel to the plate.

With a turbulent boundary layer and during heat transfer there is a thin layer of fluid in laminar flow at the surface of the plate (a laminar sub-layer) in which there is a considerable variation of velocity u and temperature t in direction of the y axis. The shearing stress and the density of heat flux at an arbitrary plane of the laminar sub-layer parallel to the plate surface can be determined from the equations

$$S_s = \mu \frac{du}{dy} , \qquad \ldots\ldots\ldots\ldots (1)$$

and

$$q_s = -K \frac{dt}{dy} , \qquad \ldots\ldots\ldots\ldots (2)$$

Assuming a linear distribution of velocity and temperature in laminar sub-layer, equations (1) and (2) can take the following forms:

$$S_s = \mu \frac{u}{y} = \mu \frac{u_s}{\delta_{h,s}} \; ; \; 0 < y < \delta_{h,s} , \quad \ldots\ldots (3)$$

and

$$q_s = K \frac{t - t_w}{y} = K \frac{t_s - t_w}{\delta_{t,s}} \; ; \; 0 < y < \delta_{t,s} \ldots (4)$$

Combining the two equations (3) and (4), we get

$$q_s = S_s . \frac{K}{\mu} . \frac{t_s - t_w}{u_s} . \frac{\delta_{h,s}}{\delta_{t,s}} \qquad \ldots\ldots\ldots (5)$$

In the turbulent part of the turbulent boundary layer, disregarding the molecular transfer of heat and momentum, the relation between the density of heat flux and the shearing stress can be regarded from the following equation

$$q_T = - S_T \cdot \frac{K_T}{\mu_T} \cdot \frac{\partial \bar{t}/\partial y}{\partial \bar{u}/\partial y}$$

$$= - S_T \cdot \frac{K}{\mu} \cdot \frac{a_T}{a} \cdot \frac{\nu}{\nu_T} \cdot \frac{\partial \bar{t}/\partial y}{\partial \bar{u}/\partial y} \quad \ldots\ldots (6)$$

Assuming that the turbulent thermal diffusivity is equal to the turbulent kinematic viscosity ($a_T = \nu_T$), the distributions of average temperature $\bar{t}$ and average velocity $\bar{u}$ in the turbulent part will be identicals. Thus, for the turbulent part of the turbulent boundary layer, equation (6) can be rewritten in the following form

$$q_T = -S_T \cdot \frac{K}{\mu} \cdot Pr \cdot \frac{\partial \bar{t}/\partial y}{\partial \bar{u}/\partial y} \quad \ldots\ldots\ldots\ldots (7)$$

As $\delta_{h,s} \ll \delta_h$, $\delta_{t,s} \ll \delta_t$ and $\delta_h \simeq \delta_t$, so we can write equation (7) as follows

$$q_T = -S_T \cdot \frac{K}{\mu} \cdot Pr \cdot \frac{t_\infty - t_s}{u_\infty - u_s} \quad \ldots\ldots\ldots\ldots (8)$$

The amount of heat transported by turbulent flow across the turbulent part of the boundary layer is equal to the amount of heat transported across the laminar sub-layer ($q_T = q_s = q$), because there is no break in the amount of heat flux at the outer edge of thermal sub-layer $y = \delta_{t,s}$. Assuming that there is no difference between the tangential stresses in equations (5) and (8), and solving these equations according to t_w and t_∞, we get

$$t_\infty - t_w = \frac{q}{S_s} \frac{1}{C_p} \left[u_\infty + u_s \left(Pr \cdot \frac{\delta_{t,s}}{\delta_{h,s}} - 1 \right) \right] \quad , \ldots\ldots (9)$$

or

$$h = \frac{q}{t_\infty - t_w} = \frac{S_s \quad C_p}{u + u_s \left(Pr.\frac{\delta_{t,s}}{\delta_{h,s}} - 1\right)} \quad \text{.......... (10)}$$

It is possible to obtain the laminar sub-layer thickness $\delta_{h,s}$ as follows: In Fig. (2) curve (1) is plotted as a linear distribution of velocity in laminar sub-layer according to equation (3), and curve (2) illustrates experimental data for the average velocity distribution in the turbulent part near the sub-layer [5]. The intersection of the two curves (1) and (2) gives a relation for calculation the thickness of the laminar sub-layer in the form

$$\delta_{h,s} \simeq 12\, \nu\, (\rho / S_s)^{1/2} \quad \text{.......... (11)}$$

From equations (3) and (11) we get

$$u_s = 12\, (S_s / \rho)^{1/2} \quad \text{.......... (12)}$$

The ratio of the thickness of the thermal sub-layer over a flat plate with non-uniform surface temperature $\delta_{t,s}$ to the thickness of the laminar sub-layer $\delta_{h,s}$ can be written in the form

$$\frac{\delta_{t,s}}{\delta_{h,s}} = \frac{\delta_{t,s}}{\delta_{t,s,c}} \times \frac{\delta_{t,s,c}}{\delta_{h,s}} , \quad \text{.......... (13)}$$

where ($\delta_{t,s}/ \delta_{t,s,c}$) is the ratio of the thickness of the thermal sub-layer over aflat plate with non-uniform surface temperature to the thickness of the thermal sub-layer over a flat plate with uniform surface temperature, and can be calculated from the following expression [1]

$$\frac{\delta_{t,s}}{\delta_{t,s,c}} = \left[\frac{\int_0^x (t_\infty - t_w)^2 \, dx}{x (t_\infty - t_w)^2} \right]^{0,5} = \varepsilon_{t,x} \, , \quad \text{......(14)}$$

and

($\delta_{t,s,c}/ \delta_{h,s}$) is the ratio of the thickness of the thermal sub-layer over a flat plate with uniform surface temperature to the thickness of laminar sub-layer, and can be determined by [5]

$$\frac{\delta_{t,s,c}}{\delta_{h,s}} = (Pr)^{-1/3} \qquad \text{.......... (15)}$$

Substituting equations (12), (13), (14) and (15) in equation (10), we obtain

$$h = \frac{S_s \cdot C_p}{u_\infty + 12\sqrt{S_s/\rho}\,(\varepsilon_{t,x} \cdot Pr^{2/3} - 1)} \qquad \text{......... (16)}$$

The shear stresses acting at the wall are characterized in hydrodynamics by the definition

$$S_s = C_f\,(\rho u_\infty^2/2). \qquad \text{........ (17)}$$

Substituting the value of S_s in equation (16) and dividing both sides of equation by $(\rho u_\infty C_p)$, we obtain a relation for Stanton number

$$St_x = \frac{Nu_x}{Re_x \cdot Pr} = \frac{C_f/2}{1 + 12\sqrt{C_f/2}\,(\varepsilon_{t,x} \cdot Pr^{2/3} - 1)} \qquad \text{..(18)}$$

The local friction coefficient of a turbulent boundary layer over a flat plate can be determined from the following expression [6]

$$C_f/2 = 0{,}0296\,(Re_x)^{-0{,}2} \qquad \text{............(19)}$$

Combining equations (18) and (19) we obtain

$$St_x = \frac{0{,}0296\,Re_x^{-0{,}2}}{1 + 2{,}064\,Re_x^{-0{,}1}(\varepsilon_{t,x} \cdot Pr^{2/3} - 1)} \qquad \text{.....(20,a)}$$

or

$$Nu_x = \frac{0{,}0296\,Re_x^{0{,}8} \cdot Pr}{1 + 2{,}064\,Re_x^{-0{,}1}(\varepsilon_{t,x} \cdot Pr^{2/3} - 1)} \qquad \text{.....(20,b)}$$

Equation (20) is the mathematical expression of heat transfer to the turbulent boundary layer over a flat plate with non-uniform surface temperature.

The relation between the heat transfer to the turbulent boundary layer over a flat plate with non-uniform surface temperature to that what would be predicted assuming quasi-uniform surface temperature can be written in the following form

$$\frac{q}{q_c} = \frac{1+2,064\ Re_x^{-0,1}\ (Pr^{2/3} - 1)}{1+2,064\ Re_x^{-0,1}\ (\varepsilon_{t,x} \cdot Pr^{2/3} - 1)} \quad ..;(21)$$

Equations (21) and (14) indicate that the ratio $(q/q_c) > 1$ as the temperature difference increases along the plate, and vice-versa.

CALCULATION AND COMPARISON WITH OTHERS EXPERIMENTAL RESULTS

Presented analytical solution is of limited usefulness unless it can be compared against an experimental data. For example, the experimental data by Moretti and Kays [4] has been chosen for comparison with the present analysis. The experimental data by Moretti and Kays for test runs with surface temperature steps at two different points and with constant free-stream velocity and temperature are shown in Fig.(3) and (4). Equation (20) is plotted for the two cases, and the results are compared with the experimental data [4] . From Figs. (3) and (4) we conclude that the present analysis is shown to agree well with the experimental data by Moretti and Kays. Also it is clear that the present analysis is more agreeabe with experimental data than the empirical modification of the Ambrok solution for step-wise variation of surface temperature [4] .

Equation (21) is plotted in Fig. (5) for the above mentioned two cases. Fig. (5) shows that the heat transfer to the turbulent boundary layer over a flat plate with a step rise of surface temperature is greater than what would be predicted assuming quasi-uniform surface temperature.

CONCLUSION

The presented solution and calculation performed show that the calculation of the local rate of the heat transfer to a turbulent boundary layer from a non-uniform temperature surface by assuming quasi-uniform conditions would involve a substantial error, particulary when surface temperature varies with a relatively high rate along the plate.

The difference between the local rate of the heat transfer from a non-uniform temperature surface and that what would be predicted assuming quasi-uniform condition gets bigger as the temperature difference is increased, and vice-versa.

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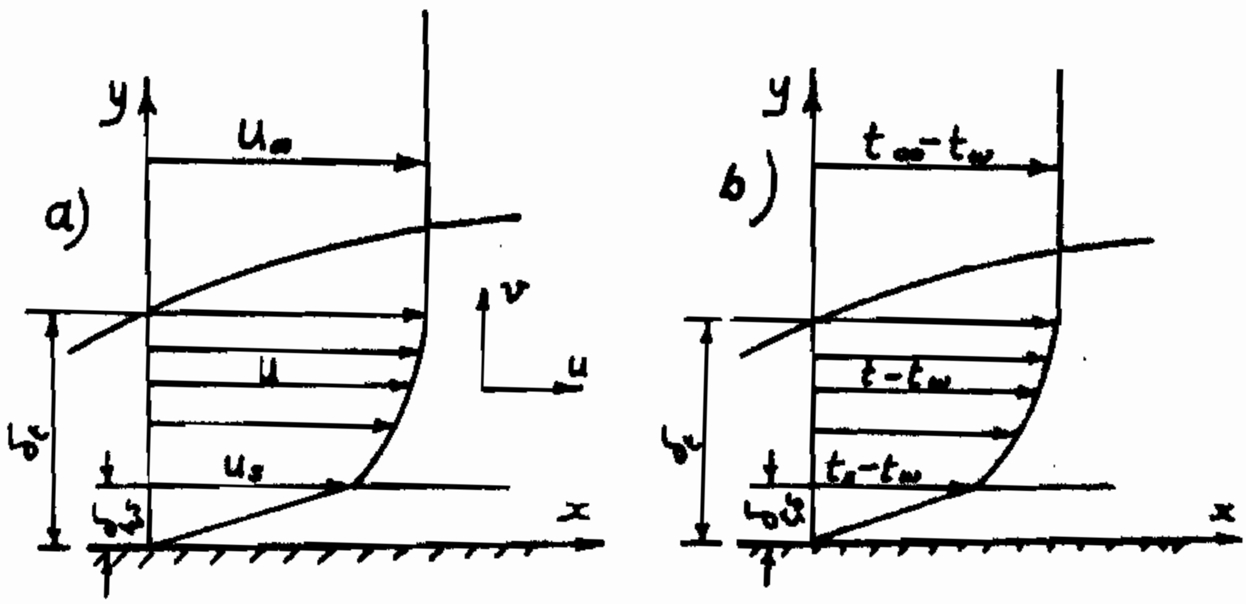

Fig. 1. Physical model and coordinate system.

a) Hydrodynamic boundary layer.   b) Thermal boundory layer.

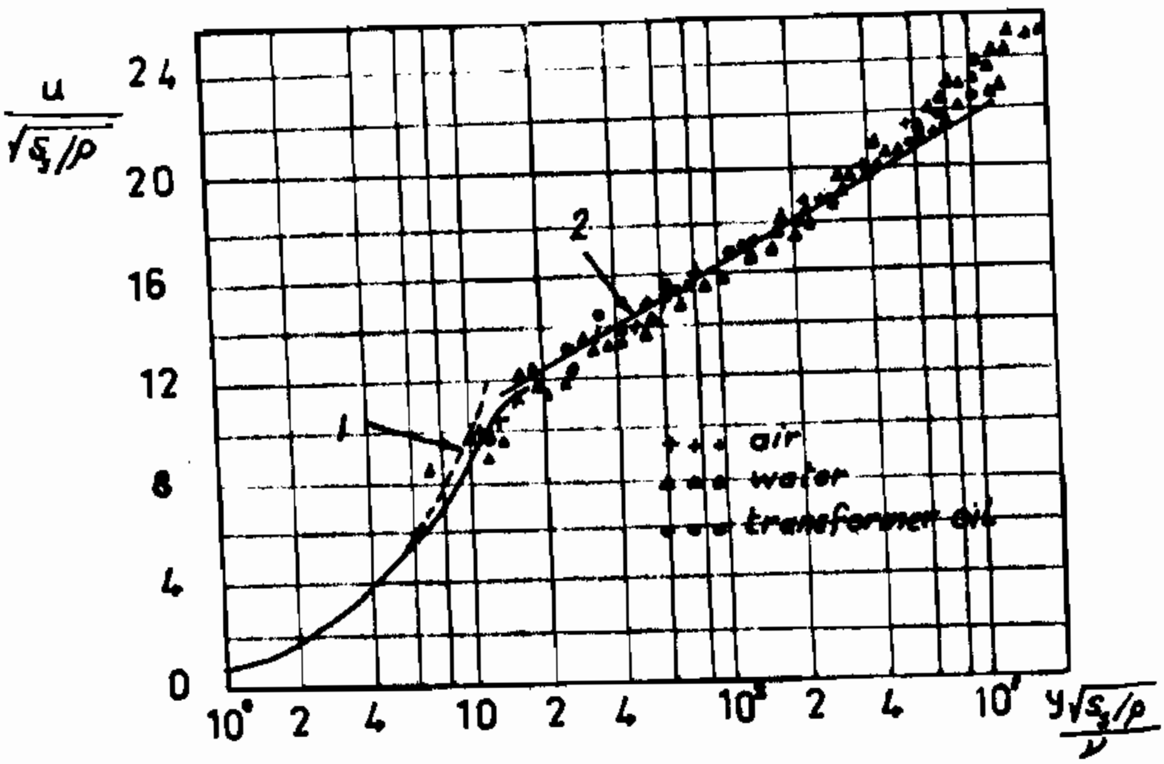

Fig. 2. Velocity distribution in a turbulent boundary layer over a flat plate

1) Velocity distribution according to Eq. (3).
2) Experimental distrtribution of average velocity in the turbulent part near the laminar sub-layer [5].

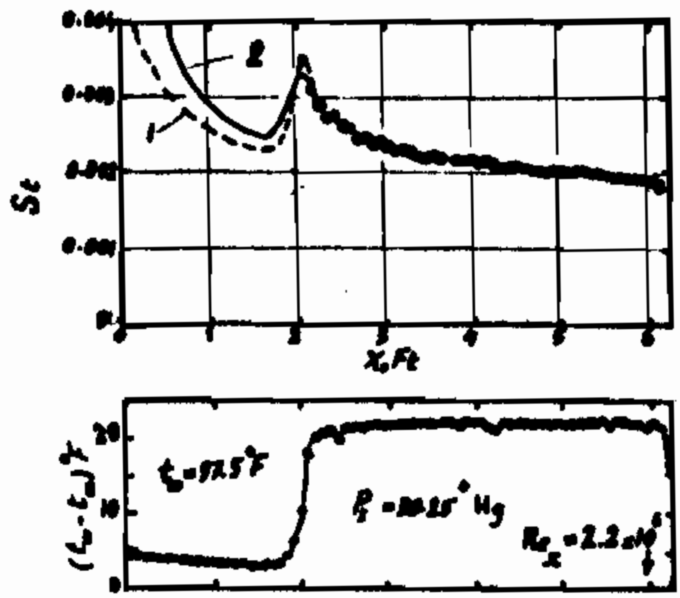

Fig. (3). Heat transfer results for late temperature step.

1- Present analysis, relation (20),

2- Empirical modification of the Ambrok solution for step variation of surface temperature [4] ,

∘∘∘ Moretti and kays experimental data [4] .

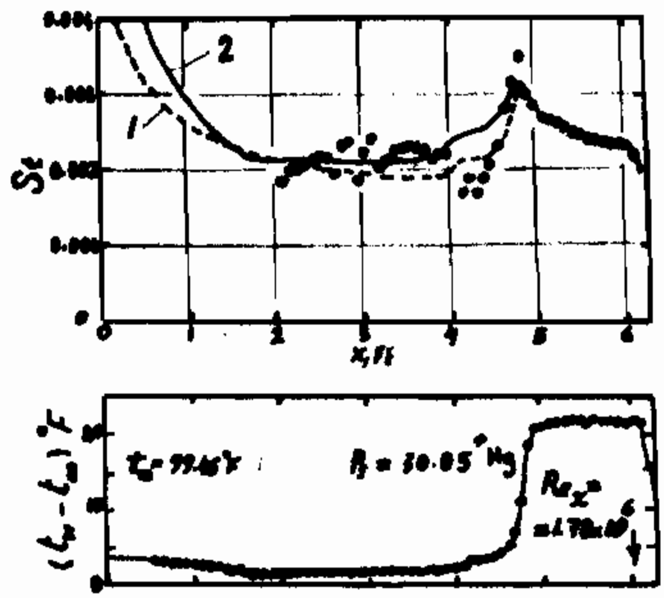

Fig. (4). Heat transfer results for early temperature step.

1- Present analysis, relation (20),

2- Empirical Modification of the Ambrok sdution for step variation of surface temperature [4],

ooo Moretti and Kays experimental data [4].

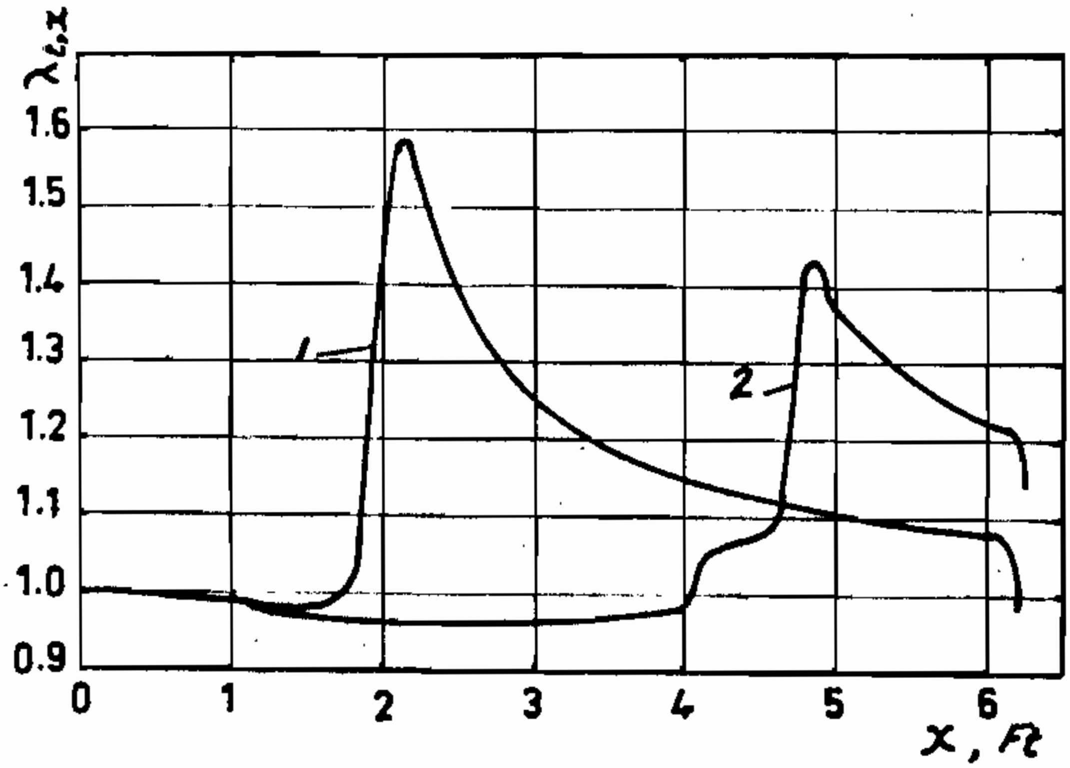

Fig. (5). Heat transfer ratio (q/q_c) for a step-wise variation of surface temperature.

1- early temperature step [Fig. (3)],
2- late temperature step [Fig. (4)].